Trails and S-graphs

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July 20, 2017

1. Basic Notation

Let $\mathfrak{g}$ be a Kac-Moody algebra.

$\{\alpha_i\}_{i\in I}$, $\{\alpha_i^\vee\}_{i\in I}$, the set of simple roots, coroots.

$\{e_i \in \mathfrak{g}\}_{i\in I}$, a choice of simple root vectors.

$\{s_i \in W\}_{i\in I}$ the set of simple reflections.

$\{\varpi_i\}_{i\in I}$ the set of fundamental weights.

2. Sequences of reduced expressions

Fix a sequence $J = (\dots, i_j, i_{j-1}, \dots, i_1) : i_j \in I$ of reduced

decompositions, that is $w_j := s_{i_j} s_{i_{j-1}} \cdots s_{i_1} \in W$ is a reduced

decomposition for all $j \in J$.

It is convenient to write $j \in J$ as (s, k), when
$s = i_j, k = |\{i_u = i_j | u \leq j\}|)$.

Fix $t \in I$ and let $V(-\varpi_t)$ denote the $\mathfrak{g}$ module with lowest weight $-\varpi_t$.

3. Trails

A trail K is a sequence of vectors $v_j^K \in V(-\varpi_t)$ of weight γ_j^K

defined by (T). For all $j \in J, \exists n_j \in \mathbb{N}$ with $e_{i_j}^{n_j} v_j^K = v_{j+1}^K$

satisfying the boundary conditions

$(B(i))$. $\gamma_j^K = -s_t\varpi_t$, for all $j \leq (t,1)+1$.

$(B(ii))$. $\gamma_{j+1}^K = -w_j\varpi_t$, for all $j >> 0$.

The set of all (Berenstein-Zelevinsky) trails is denoted $\mathscr{K}_t^{BZ}$.

4. Functions

Set $\delta_j^K = \frac{1}{2}(\gamma_j^K + \gamma_{j+1}^K)$.

Set

$$z^K = \sum_{j \in J} \alpha_{i_j}^\vee(\delta_j^K) m_j,$$

where the $\{m_j\}_{j \in J}$ are viewed as co-ordinate functions.

Note that z^K determines the trail K.

A trail K is said to trivialize at w_j if $\gamma_{k+1}^K = -w_k \varpi_t : \forall k \geq j$.

The unique trail K_t^1 which trivializes at $(t, 1)$ is called the driving trail.

5. The Kashiwara Crystal

The set B_J is defined by giving the co-ordinate functions $\{m_j\}_{j\in J}$ non-negative integer values, almost all zero.
Write $m_j = m_s^k$, when $j \in J$ is written as $(s,k) \in I \times \mathbb{N}^+$.

Kashiwara gave B_J a crystal structure through the Kashiwara functions

$$r_s^k = m_s^k + \sum_{j>(s,k)} \alpha_{i_j}^\vee(\alpha_s) m_j, \forall (s,k) \in I \times \mathbb{N}^+.$$

These functions describe how the Kashiwara operators $\tilde{e}_i, \tilde{f}_i : i \in I$ act on B_J.

In particular the Kashiwara parameters $\{\varepsilon_i\}_{i\in I}$ are defined by

$$\varepsilon_i(b) = \max_{k\in\mathbb{N}^+} r_i^k(b), \forall b \in B_J.$$

6. The Kashiwara-Verma subcrystal $B(\infty)$

$B_J(\infty)$ is defined as the subcrystal of B_J generated by the zero vector.

Notably as a crystal it is independent of J and denoted by $B(\infty)$.

Again for all $i \in I, b \in B(\infty)$, the value $\varepsilon_i(b)$ is just the largest value of $k \in \mathbb{N}$ such that $\tilde{e}_i^k b \neq 0$.

$B(\infty)$ has a rich combinatorial structure and in particular determines the subcrystals corresponding to all the maximal simple integrable quotients of Verma modules (which are simple if $\mathfrak{g}$ is symmetrizable).

7. The Fundamental Problems

Problem 1. Describe $B_J(\infty)$ as a subset of B_J.

Fact: $B_J(\infty)$ admits a bijection $\star$, which in the non-symmetrizable case in not easy to construct. It makes no sense to ask if $\star$ is linear since it is not defined on B_J, nor obvious how it could be extended to B_J.

Yet as a consequence $B_J(\infty)$ admits sets $Z_t : t \in I$ of dual Kashiwara functions by transport of structure and they define dual Kashiwara parameters by

$$\varepsilon_t^\star(b) = \max_{z \in Z_t} z(b) : b \in B_J.$$

One can use these dual Kashiwara parameters, **if** they can be computed, to describe $B_J(\infty)$.

8. $\mathfrak{g}$ is semisimple

After Berenstein-Zelevinsky we can take $Z_t = \{z^K : K \in \mathscr{K}_t^{BZ}\}$, which are linear functions on B_J.
As a consequence one can show that $B_J(\infty)$ is a polyhedral subset of B_J.
Nevertheless trails are not combinatorially defined and almost impossible to compute. Thus one cannot describe this polyhedral subset.

Problem 2. Describe $\mathscr{K}_t^{BZ}$ combinatorially.

Problem 3. Extend the BZ result to all $\mathfrak{g}$ Kac-Moody.

The latter in particular needs another approach because GP and BZ need that the choices of J form a single orbit under Coxeter moves. Already this fails for the affinisation of A_1.

9. Invariance Properties

The dual Kashiwara parameters are almost invariant under the Kashiwara crystal operators.

This invariance property leads to the notion of an S-graph.

An S-graph is a finite graph $\mathscr{G}$ with vertices labelled by $\hat{N} = \{1, 2, \dots, n\}$, for some $n \in \mathbb{N}^+$, satisfying notably the S property below.

Let $V(\mathscr{G})$ denote the set of vertices of $\mathscr{G}$.

For all $k \in \hat{N}$, let $V^k(\mathscr{G})$ denote the subset of $V(\mathscr{G})$ of vertices with label k.

The crucial S property is the following.

For every $v \in V(\mathscr{G})$ and every $k \in \hat{N}$ there is a vertex $v' \in V^k(\mathscr{G})$ and an ordered path from v to v'.

S-graphs seem pretty fundamental except that they do not exist if an ordered path refers to following arrows on edges!

10. S-graphs

Set $N = \{1, 2, \dots, n-1\}$. An S-graph is defined relative to some $s \in I$ and a coefficient set $\mathbf{c} = \{c_i\}_{i \in N}$ of non-negative integers. The edges of an S-graph are labelled by the elements of $\mathbf{c}$. *Then an ordered path just means that coefficients increase along the edges.*
With this slight modification S-graphs exist!
S-graphs are required to have several additional properties.

A crucial one is **evaluation**, defined by
To each vertex v a function f_v is assigned and these satisfy

$$f_v - f_{v'} = c_{v,v'}(r_s^{i_v} - r_s^{i_{v'}}),$$

where v, v' are adjacent vertices with labels $i_v, i_{v'}$ joined by the edge with label $c_{v,v'}$.

11. S-sets

An S-graph is defined by a "driving function" of type $s \in I$. This provides s, n and coefficient set $\mathbf{c}$.

To encode this data an S graph is assume to admit a unique pointed chain $\{v_1, v_2, \dots, v_n\}$ with $v_i \in V^i(\mathscr{G})$ and joined to v_{i+1} by an edge with label c_i.

Thus v_n is a distinguished element of the vertex set and f_{v_n} is assigned the driving function.

An S-graph is assumed connected.

An S-set is the set $\{f_v\}_{v \in V(\mathscr{G})}$.

12. Canonical S-graphs

Even with additional conditions an S-graph given by a coefficient set $\mathbf{c}$ is not uniquely determined. However there is a "canonical" S-graph $\mathscr{G}(\mathbf{c})$ for each choice of $\mathbf{c}$. $\mathscr{G}(\mathbf{c})$ has some remarkable special properties.

Assume for simplicity that $c_i > 0$, for all $i \in N$. Then $\mathscr{G}(\mathbf{c})$ has the structure of an n-dimensional hypercube with exactly no edges joining vertices with the same index.

When the c_i are pairwise distinct, the $f_v : v \in V(\mathscr{G})$ are pairwise distinct.

To preserve the above property in general, $\mathscr{G}(\mathbf{c})$ may be contracted by identifying adjacent vertices with the same index and the same attached function.

The above construction replaces squares by triangles. Eventually when all the coefficients are equal, $\mathscr{G}(\mathbf{c})$ degenerates to the n-simplex.

13. Binary Fusion

The order relation on N induced by the natural order on $\mathbf{c}$ is lifted to a total order.
Then the canonical S graphs are obtained by induction on n using a process of "binary fusion".
On the other hand there are graphs $\mathscr{G}_n$ whose vertices which are equivalence classes of (unordered) partitions of $n+1$ with boundary conditions and edges defined by "single block linkages". The number of vertices of $\mathscr{G}_n$ is the Catalan number $C(n) = \frac{1}{n+1}\binom{2n}{n}$.
$\mathscr{G}_n$ admits a "natural" evaluation map, that is an assignment of a function to each vertex v of $\mathscr{G}_n$.
The $\mathscr{G}(\mathbf{c})$ can be characterized as the unique subgraphs of $\mathscr{G}_n$ in which the linear order is encoded by "triads" in $\mathscr{G}_n$.
The $\mathscr{G}(\mathbf{c})$ are determined by just the linear order on N. Yet the number of distinct graph is less than $(n-1)!$ and indeed the Catalan number $C(n-1)$.
By studying the degeneration of the hypercubes described above, one shows that the canonical S-set obtained from $\mathbf{c}$ is independent of the lifting of the order relation on $\mathbf{c}$.

14. Convexity

The S-sets obtained from the canonical graphs have some important additional properties.
For any choice of $s \in I$, view the successive differences $\{r_s^i - r_s^{i+1}\}_{i \in N}$ as co-ordinate functions on $\mathbb{Q}^n$. Then the S-set $Z(\mathbf{c})$ defined by $\mathscr{G}(\mathbf{c})$ form the extremal elements of a convex set $K(\mathbf{c})$ in $\mathbb{Q}^n$.
The set $K(\mathbf{c})$ is relatively easy to define. From it one may deduce $Z(\mathbf{c})$ and this can be a convenient way to compute the latter, rather than going through the inductive binary fusion construction.

The very simplest case is when the $\{c_i\}_{i \in N}$ are increasing.
In this case $K(\mathbf{c}) = (c_i')_{i \in N}$ where the $\{c_i'\}_{i \in N}$ are increasing and $0 \leq c_i' \leq c_i$, for all $i \in N$.

15. Parametrization of Trails

Take an element $K \in \mathscr{K}_t^{BZ}$ which trivializes at $w := w_j$. Write $j \in J$ as (s, n). Set $e_s = e$. Let f denote the image of e under a Chevalley anti-automorphism. Together they generate the copy of $\mathfrak{sl}(2)$ used below. Let $v_{\mathbf{k}}$ denote the vector $e^{k_n} v_{-a_n} \otimes e^{k_{n-1}} v_{-a_{n-1}} \otimes \cdots \otimes e^{k_1} v_{-a_1}$ in the n-fold tensor product $V(-a_n) \otimes V(-a_{n-1}) \otimes \cdots \otimes V(-a_1)$ of lowest weight $\mathfrak{sl}(2)$ modules.
Let $e_{-a_j} : j > 1$ is a product of simple root vectors distinct from e and of weight $-a_j$.
Since $(\text{ad}\, f)e_{-a_j} = 0$ we may write $v_{-w\varpi_t}$ as the $\mathfrak{sl}(2)$ module image $\overline{v}_{\mathbf{k}}$ of $v_{\mathbf{k}}$ given by

$$\overline{v}_{\mathbf{k}} := e^{k_n} e_{-a_n} e^{k_{n-1}} e_{-a_{n-1}} \cdots e^{k_1} v_{-a_1}.$$

We call $\overline{v}_{\mathbf{k}}$ a monomial expression.
This image is non-trivial on account of the Chevalley-Serre relations which imply that $(\text{ad}\, e)^{a_j+1} e_{a_j} = 0$.
Let $T_s(\mathbf{a})$ denote the set of all such *non-zero* monomial expressions with the $e_{-a_j} : j > 1$ fixed.

16. Adjoining a Face

Fix $s \in I, k \in \mathbb{N}$. Then $r_s^k - r_s^{k+1}$ is defined to be the face function $z^{F_s^{k+1}}$ given by the "face" F_s^{k+1}.

We say that a face F_s^{k+1} may be adjoined to a trail $K \in \mathscr{K}_t^{KZ}$ if $K + F_s^{k+1} \in \mathscr{K}_t^{BZ}$ and $z^{K+F_s^{k+1}} = z^K + z^{F_s^{k+1}}$.

Adjoining the face F_s^{k+1} is the operation of moving a factor of e from its place at $(s, k+1)$ to (s, k).

F_s^{k+1} can be visualized as a genuine "face" through wiring diagrams.

It is far from obvious when adjoining a face is possible.

Problem 4. Show that $\mathscr{K}_t^{BZ}$ is generated by adjoining faces to the driving trail K_t^1.

This is closely analogous to generating the lowest weight crystal $B(-\varpi_t)$ by the Kashiwara crystal operators.

17. The Matching Condition

The possibility of adjoining a face to a trail depends crucially on a "matching condition" which ensures that moving e to the right through $e_{-a_{k+1}}$ only changes $\overline{v}_{\mathbf{k}}$ by a (non-zero) scalar.

At first sight it might seem that such a property could only hold by a miracle. In fact it results from the boundary condition $B(ii)$ and an elementary property of Demazure modules, namely $(*)$ below.

Lemma

The $\mathfrak{sl}(2)$ module M spanned by the elements of $T_s(\mathbf{a})$ is simple.

18. The Matching Condition. The proof

Proof.

Recall that $w \in W$ has reduced decomposition $w = s_{i_j} s_{i_{j-1}} \cdots s_{i_1}$.
Set $V_w(-\varpi_t) := U(\mathfrak{n}^-) v_{-w\varpi_t}$.
One has

$$V_w(-\varpi_t) = \mathbb{C}[e_{i_j}]\mathbb{C}[e_{i_{j-1}}] \cdots \mathbb{C}[e_{i_1}] v_{-\varpi_t}. \qquad (*)$$

Thus $v \in M$ belongs to $U(\mathfrak{b}^-) v_{-w\varpi_t}$. Moreover if v is a weight vector then its weight must lie in $-w\varpi_t + \mathbb{Z}\alpha_s$, by definition of $T_s(\mathbf{a})$. Yet the only weight vectors of weight $\mathbb{Z}\alpha_s$ lying in $U(\mathfrak{n}^-)$ are the powers of f_s. Thus $v = f_s^n v_{-w\varpi_t}$ for some $n \in \mathbb{N}$, from which the simplicity of M follows. $\square$

N.B. Unfortunately this says nothing about the possible vanishing of monomial expressions.

19. Minimal Trails

A trail $K \in T_s(\mathbf{a})$ is given by an n-tuple $\mathbf{k} = (k_n, k_{n-1}, \dots, k_1)$.

The minimal trail $K_{\ell \min} \in T_s(\mathbf{a})$ is the unique minimal element under lexicographic order on n-tuples.

Set $c_i = -\alpha_i^\vee(\delta_{(s,i)}^{K_{\ell \min}})$, for all $i \in \mathbb{N}^+$.

Lemma

$c_i \in \mathbb{N}$, for all $i \in \mathbb{N}^+$ and vanishes when $i \geq j - 1$.

The proof uses the Chevalley-Serre relations.

Thus a minimal trail gives the initial data $\{c_i\}_{i \in N}$ of an S-graph with $z^{K_{\ell \min}}$ being the driving function.

20. Absence of False Trails

Identify the minimal trail in $T_s(\mathbf{a})$ with the n-tuple $\mathbf{l}$.
Problem 5. Show that $T_s(\mathbf{a})$ identifies with the set $K_{\mathbb{Z}}(\mathbf{c})$ of integer points of $K(\mathbf{c})$ by taking $c_i' = k^{(i)} - \ell^{(i)}$.
Problems $1-5$ are settled if there are no "false trails". Moreover all trails are hereby defined purely combinatorially and fairly explicitly.
There being no false trails can be expressed as

$$T_s(\mathbf{a}) \subset K_{\mathbb{Z}}(\mathbf{a}), \tag{1}$$

for all possible choices.
Our goal is to prove $(*)$ by induction. Let $T_s^-(\mathbf{a})$ the subset of $T_s(\mathbf{a})$ of trails which trivialize at w_{j-1} and $K_{\mathbb{Z}}^-(\mathbf{c})$ the subset of all $\mathbf{c}' \in K_{\mathbb{Z}}(\mathbf{c})$ such that $c_{n-1}' = 0$. It is trivial that (1) implies

$$T_s^-(\mathbf{a}) \subset K_{\mathbb{Z}}^-(\mathbf{c}). \tag{1'}$$

Theorem

$(1')$ *implies* (1).

Unfortunately this is not quite the end of the story since the notion of a false trail depends on the choice of some $s \in I$.

21. The Rigid Case

We give the proof of the theorem in the easy case when the c_i are increasing.
The first step is to compute the coefficient of $v_{\mathbf{l}}$ in $f^b v_{\mathbf{k}}$, where $b_i := k_i - \ell_i \geq 0$ and $b = \sum (k_i - \ell_i)$.
Up to a positive factor it is

$$\prod_{j=1}^{r} \prod_{i=1}^{b_j} (i + k^{(j-1)} + \ell^{(j)} - 1 - a^{(j)}), \qquad (2)$$

where $a^{(j)} = \sum_{i=1}^{j} a_i$, etc.
From the S-graph defined when the c_i are increasing and hypothesis $(1')$, it follows that $v_{\mathbf{l}}$ is the only monomial presentation of the lowest weight vector. This forces $k_i - \ell_i \geq 0$ and hence that the c_i' are increasing. Again by uniqueness of presentation one should not be able to push e through $e_{-a_{j+1}}$ and so through the Chevalley-Serre relations. Thus one must have

$$a_{j+1} \geq \ell_{j+1} + \ell_j, \forall j = 1, 2, \ldots, n-2. \qquad (3)$$

Independently (3) is equivalent to the c_i being increasing. Uniqueness of presentation is why we call it the rigid case.

Lemma

The condition $a^{(j)} - k^{(j)} - \ell^{(j-1)} \geq 0$, *for all* $j \in \hat{N}$ *implies that the coefficient of* $v_{\mathbf{l}}$ *in* $f^b v_{\mathbf{k}}$ *is non-zero.*

Proof.

Inspect formula (2). □

Proposition

In the rigid case and under the hypothesis (1) *one has* $a^{(j)} - k^{(j)} - \ell^{(j-1)} \geq 0$, *for all* $j \in \hat{N}$.

23. Proof

Proof.

We first show that

$$a^{(j)} - k^{(j)} - \ell^{(j-1)} \geq 0, \tag{4}$$

by induction on j.
It is trivial for $j = 1$.
Combining (3) and (4) we obtain

$$a^{(j+1)} - k^{(j)} - \ell^{(j+1)} \geq 0, \tag{5}$$

On the other hand the factors $a^{(j+1)} + 1 - i - k^{(j)} - \ell^{(j+1)}$ occurring in (2) decrease in i, are integer and for $i = 1$, this factor is non-negative by (5). Recall that $b_{j+1} = k_{j+1} - \ell_{j+1}$. By uniqueness of presentation the expression in (2) must be non-zero. thus the above factors must be positive for all $i \in [1, b_{j+1}]$. Taking $i = b_{j+1}$ recovers (4) with j increased by 1. Hence the assertion.

□