On Finite W-algebras for Lie Superalgebras

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Joint work with

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Research Workshop of Israel Science Foundation

Canicular days

University of Haifa, July 21, 2017

1. Introduction

• A *finite W-algebra* is a certain associative algebra attached to a pair $(\mathfrak{g}, e)$ where $\mathfrak{g}$ is a complex semi-simple Lie algebra and $e \in \mathfrak{g}$ is a nilpotent element.

• A finite W-algebra is a generalization of the universal enveloping algebra $U(\mathfrak{g})$. For $e = 0$ it coincides with $U(\mathfrak{g})$.

• Finite W-algebra is a quantization of the Poisson algebra of functions on the Slodowy (i.e. transversal) slice at e to the orbit $Ad(G)e$, where $\mathfrak{g} = Lie(G)$.

• Due to recent results of I. Losev, A. Premet and others, finite W-algebras play a very important role in description of primitive ideals.

• Finite W-algebras for semi-simple Lie algebras were introduced by A. Premet.

• Finite W-algebras for Lie algebras and superalgebras have been studied by mathematicians and physicists: L. Fehér, C. Briot, E. Ragoucy, P. Sorba, A. Premet, I. Losev, V. Ginzburg, W. L. Gan, J. Brundan, A. Kleshchev, J. Brown, S. Goodwin, W. Wang, L. Zhao, Y. Zeng, B. Shu, Y. Peng.

2. Finite W-algebras for Lie superalgebras

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra with reductive even part $\mathfrak{g}_{\bar{0}}$.

Let $\chi \in \mathfrak{g}_{\bar{0}}^* \subset \mathfrak{g}^*$ be an even nilpotent element in the coadjoint representation, i.e. the closure of the $G_{\bar{0}}$-orbit of χ in $\mathfrak{g}_{\bar{0}}^*$ contains zero. ($G_{\bar{0}}$ is the algebraic reductive group of $\mathfrak{g}_{\bar{0}}$.)

Definition. *The annihilator of* χ *in* $\mathfrak{g}$ is

$$\mathfrak{g}^\chi = \{x \in \mathfrak{g} \mid \chi([x, \mathfrak{g}]) = 0\}.$$

Definition. *A good* $\mathbb{Z}$*-grading for* χ is a $\mathbb{Z}$-grading $\mathfrak{g} = \oplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ satisfying the following two conditions:

(1) $\chi(\mathfrak{g}_j) = 0$ if $j \neq -2$,

(2) $\mathfrak{g}^\chi$ belongs to $\bigoplus_{j \geq 0} \mathfrak{g}_j$.

• $\chi([\cdot,\cdot])$ defines a non-degenerate skew-symmetric even bilinear form on $\mathfrak{g}_{-1}$.

Let $\mathfrak{l}$ be a maximal isotropic subspace with respect to this form.

$\mathfrak{m} := (\oplus_{j\leq -2}\mathfrak{g}_j) \bigoplus \mathfrak{l}$ is a nilpotent subalgebra of $\mathfrak{g}$.

The restriction of χ to $\mathfrak{m}$,

$$\chi : \mathfrak{m} \longrightarrow \mathbb{C}$$

defines a one-dimensional representation $\mathbb{C}_\chi =< v >$ of $\mathfrak{m}$.

Let I_χ be the left ideal of $U(\mathfrak{g})$ generated by $a - \chi(a)$ for all $a \in \mathfrak{m}$.

Definition. The induced $\mathfrak{g}$-module

$$Q_\chi := U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} \mathbb{C}_\chi \cong U(\mathfrak{g})/I_\chi$$

is called *the generalized Whittaker module.*

Definition. *The finite W-algebra* associated to the nilpotent element χ is

$$W_\chi := \mathrm{End}_{U(\mathfrak{g})}(Q_\chi)^{op}.$$

• As in the Lie algebra case, the superalgebras W_χ are all isomorphic for different choices of good $\mathbb{Z}$-gradings and maximal isotropic subspaces $\mathfrak{l}$.

• By Frobenius reciprocity

$$\mathrm{End}_{U(\mathfrak{g})}(Q_\chi) = \mathrm{Hom}_{U(\mathfrak{m})}(\mathbb{C}_\chi, Q_\chi).$$

That defines an identification of W_χ with the subspace

$$Q_\chi^{\mathfrak{m}} = \{u \in Q_\chi \mid au = \chi(a)u \text{ for all } a \in \mathfrak{m}\}.$$

- Let $\pi : U(\mathfrak{g}) \to U(\mathfrak{g})/I_\chi$ be the natural projection. Then

$$W_\chi = \{\pi(y) \in U(\mathfrak{g})/I_\chi \mid (a - \chi(a))y \in I_\chi \text{ for all } a \in \mathfrak{m}\},$$

Equivalently,

$$W_\chi = \{\pi(y) \in U(\mathfrak{g})/I_\chi \mid \mathrm{ad}(a)y \in I_\chi \text{ for all } a \in \mathfrak{m}\}.$$

The algebra structure on W_χ is given by

$$\pi(y_1)\pi(y_2) = \pi(y_1 y_2)$$

for $y_i \in U(\mathfrak{g})$ such that $\mathrm{ad}(a)y_i \in I_\chi$ for all $a \in \mathfrak{m}$ and $i = 1, 2$.

• The case of an *even* good $\mathbb{Z}$-grading is easier!

Definition. A $\mathbb{Z}$-grading $\mathfrak{g} = \oplus_{j \in \mathfrak{g}} \mathfrak{g}_j$ is called *even*, if $\mathfrak{g}_j = 0$ unless j is an even integer.

Let $\mathfrak{p} := \oplus_{j \geq 0} \mathfrak{g}_j$ be a parabolic subalgebra of $\mathfrak{g}$. Then

$$W_\chi = U(\mathfrak{p})^{\mathfrak{m}} := \{y \in U(\mathfrak{p}) \mid [a, y] \in I_\chi \text{ for all } a \in \mathfrak{m}\}.$$

Remark. If $\mathfrak{g}$ admits an even non-degenerate $\mathfrak{g}$-invariant supersymmetric bilinear form, then $\mathfrak{g} \simeq \mathfrak{g}^*$ and $\chi(x) = (e|x)$ for some nilpotent $e \in \mathfrak{g}_{\bar{0}}$ (i.e. ad$e$ is a nilpotent endomorphism of $\mathfrak{g}$).

e can be included in $\mathfrak{sl}(2) =< e, h, f >\subset \mathfrak{g}_{\bar{0}}$ by the Jacobson-Morozov theorem.

adh defines a Dynkin $\mathbb{Z}$-grading $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_j$, which is good for χ.

- Good $\mathbb{Z}$-grading for basic Lie superalgebras were classified by C. Hoyt *(Israel J. Math. 2012)*.

Example. Let $e = 0$. Then $\chi = 0$, $\mathfrak{g}_0 = \mathfrak{g}, \mathfrak{m} = 0$,

$$Q_\chi = U(\mathfrak{g}), \quad W_\chi = U(\mathfrak{g}).$$

Let $\mathfrak{g}^e = \mathrm{Ker}(ade)$. Then $\mathfrak{g}^e = \mathfrak{g}^\chi$, $\dim \mathfrak{g}^e = \dim \mathfrak{g}_0 + \dim \mathfrak{g}_1$.

Definition. A nilpotent $\chi \in \mathfrak{g}_{\bar{0}}^*$ is called *regular* nilpotent if $G_{\bar{0}}$-orbit of χ has maximal dimension, i.e. the dimension of $\mathfrak{g}_{\bar{0}}^\chi$ is minimal.
Equivalently, a nilpotent $e \in \mathfrak{g}_{\bar{0}}$ is *regular* nilpotent, if the centralizer $\mathfrak{g}_{\bar{0}}^e$ attains the minimal dimension, which is equal to rank$\mathfrak{g}_{\bar{0}}$.

Example. $\mathfrak{g} = \mathfrak{sl}(n)$.

$e \in \mathfrak{sl}(n)$ is nilpotent if and only if e is an $n \times n$-matrix with eigenvalues zero.

e is a **regular** nilpotent $\Longleftrightarrow$ its Jordan normal form contains a single Jordan block

$$e = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Theorem. *(B. Kostant, Invent. Math. 1978)*
For a reductive Lie algebra $\mathfrak{g}$ and a *regular* nilpotent element $e \in \mathfrak{g}$, the finite W-algebra W_χ is isomorphic to the center of $U(\mathfrak{g})$.

- This theorem does not hold for Lie superalgebras, since W_χ must have a non-trivial odd part, and the center of $U(\mathfrak{g})$ is even.

Definition. *Kazhdan filtration* on W_χ.

Define the $\mathbb{Z}$-grading on $T(\mathfrak{g})$ induced by the shift by 2 of the fixed good $\mathbb{Z}$-grading.

For $X \in \mathfrak{g}_j$ set

$$\deg X = j + 2.$$

This induces a filtration on $U(\mathfrak{g})$, and therefore on $U(\mathfrak{g})/I_\chi$ and on $W_\chi \subset U(\mathfrak{g})/I_\chi$.

Theorem. (*A. Premet, Adv. Math. 2002*)

Let $\mathfrak{g}$ be a semi-simple Lie algebra. Then

the associated graded algebra $Gr(W_\chi)$ is isomorphic to $S(\mathfrak{g}^\chi)$;

the center of W_χ coincides with the center of $U(\mathfrak{g})$.

3. Premet's theorem for Lie superalgebras

• Let $\mathfrak{l}$ be a Lagrangian subspace in $\mathfrak{g}_{-1}$, and let $\mathfrak{l}'$ be some subspace in $\mathfrak{g}_{-1}$ satisfying the following two properties:

(1) $\mathfrak{g}_{-1} = \mathfrak{l} \oplus \mathfrak{l}'$

(2) $\mathfrak{l}'$ contains a maximal isotropic subspace with respect to the form $\chi([\cdot, \cdot])$ on $\mathfrak{g}_{-1}$.

If $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ is **even**, then $\mathfrak{l}'$ is a maximal isotropic subspace.

If $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ is **odd**, then $\mathfrak{l}^{\perp} \cap \mathfrak{l}'$ is one-dimensional and we fix **(an odd)** $\theta \in \mathfrak{l}^{\perp} \cap \mathfrak{l}'$ such that $\chi([\theta, \theta]) = 2$. Then $\pi(\theta) \in W_{\chi}$.

Conjecture.

Assume that $\mathfrak{g}$ is a Lie superalgebra with reductive even part $\mathfrak{g}_{\bar{0}}$.

If $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ is **even**, then $Gr_K W_\chi \simeq S(\mathfrak{g}^\chi)$

If $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ is **odd**, then $Gr_K W_\chi \simeq S(\mathfrak{g}^\chi) \otimes \mathbb{C}[\xi]$,

where $\mathbb{C}[\xi]$ is the exterior algebra generated by one element ξ.

• Y. Zheng and B. Shu proved the PBW theorem for finite W-algebras for basic Lie superalgebras over $\mathbb{C}$ of any type except $D(2,1;\alpha)$, where $\alpha \notin \bar{\mathbb{Q}}$ *(J. Algebra, 2015)*.
They considered two cases depending on the parity of $\dim(\mathfrak{g}_{-1})_{\bar{1}}$.
As a Corollary they proved this Conjecture.

• We proved that if χ is regular nilpotent, and
$\mathfrak{g} = D(2,1;\alpha)$, then $Gr_K W_\chi \simeq S(\mathfrak{g}^\chi) \otimes \mathbb{C}[\xi]$.
(*E.P., J. Math. Phys. 2016*).

4. Finite W-algebra for $\mathfrak{gl}(m|n)$

• E. Ragoucy and P. Sorba first observed that in the case when $\mathfrak{g}$ is the general linear Lie algebra and e consists of n Jordan blocks each of size l, the finite W-algebra for $\mathfrak{g}$ is isomorphic to the truncated Yangian of level l associated to $\mathfrak{gl}(n)$, which is a certain quotient of the Yangian Y_n for $\mathfrak{gl}(n)$.
(Comm. Math. Phys. 1999).

• J. Brundan and A. Kleshchev generalized this result to an arbitrary nilpotent e, and obtained a realization of the finite W-algebra for the general linear Lie algebra as a quotient of a so-called shifted Yangian.
(Adv. Math. 2006).

• J. Brown, J. Brundan and S. Goodwin proved that the finite W-algebra for $\mathfrak{g} = \mathfrak{gl}(m|n)$ associated to **regular (principal)** nilpotent element is a certain truncation of a shifted version of the super-Yangian $Y(\mathfrak{gl}(1|1))$.

They also proved that all irreducible modules over this algebra are finite-dimensional and classified them by highest weight theory *(Algebra Number Theory, 2013)*.

5. The super-Yangian of $\mathfrak{gl}(1|1)$

$$\mathfrak{gl}(1|1) = \{A = \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right) \mid a,b,c,d \in \mathbb{C}\} \qquad [A,B] = AB - (-1)^{p(A)p(B)} BA$$

Definition. The *super-Yangian* $Y_{1|1} = Y(\mathfrak{gl}(1|1))$ is an associative unital superalgebra over $\mathbb{C}$ with a countable set of generators

$$T_{i,j}^{(r)} \text{ where } i,j = 1,2, \text{ and } r \geq 0.$$

The $\mathbb{Z}_2$-grading of $Y_{1|1}$ is defined by

$$p(T_{i,j}^{(r)}) = p(i) + p(j).$$

We employ the formal series:

$$T_{i,j}(u) = \sum_{r \geq 0} T_{i,j}^{(r)} u^{-r} \in Y_{1|1}[[u^{-1}]].$$

- Relations in $Y_{1|1}$:

$$(u-v)[T_{i,j}(u), T_{k,l}(v)] = (-1)^{p(i)p(k)+p(i)p(l)+p(k)p(l)}((T_{k,j}(u)T_{i,l}(v) - T_{k,j}(v)T_{i,l}(u)).$$

- The **evaluation homomorphism** $\mathrm{ev}: Y_{1|1} \to U(\mathfrak{gl}(1|1))$ is defined by

$$\mathrm{ev}(T_{i,j}^{(r)}) = \begin{cases} (-1)^{p(i)} e_{i,j} \text{ if } r = 1, \\ 0 \text{ if } r > 1 \end{cases}$$

- $Y_{1|1}$ is a **Hopf algebra** with **comultiplication** given by

$$\Delta(T_{i,j}^{(r)}) = \sum_{s=0}^{r} \sum_{k} T_{i,k}^{(s)} \otimes T_{k,j}^{(r-s)}.$$

- **Gauss factorization:**

$$T(u) := \begin{pmatrix} T_{1,1}(u) & T_{1,2}(u) \\ T_{2,1}(u) & T_{2,2}(u) \end{pmatrix} = F(u)D(u)E(u)$$

$$D(u) = \begin{pmatrix} d_1(u) & 0 \\ 0 & d_2(u) \end{pmatrix}, \quad E(u) = \begin{pmatrix} 1 & e(u) \\ 0 & 1 \end{pmatrix}, \quad F(u) = \begin{pmatrix} 1 & 0 \\ f(u) & 1 \end{pmatrix}$$

$$d_i(u) = \sum_{r\geq 0} d_i^{(r)} u^{-r}, \quad e(u) = \sum_{r\geq 1} e^{(r)} u^{-r}, \quad f(u) = \sum_{r\geq 1} f^{(r)} u^{-r}$$

- **Drinfeld generators:** $Y_{1|1}$ is generated by even elements $d_1^{(r)}, d_2^{(r)}$ for $r > 0$, and odd elements $e^{(r)}, f^{(r)}$ for $r > 0$.

6. Shifted super-Yangian $Y_{1|1}(\sigma)$

Let

$$\sigma = \begin{pmatrix} 0 & s_{1,2} \\ s_{2,1} & 0 \end{pmatrix}, \text{ where } s_{1,2}, s_{2,1} \geq 0 \text{ are integers}$$

Definition. $Y_{1|1}(\sigma)$ is a subalgebra of $Y_{1|1}$ generated by $d_1^{(r)}, d_2^{(r)}$ for $r > 0$, $e^{(r)}$ for $r > s_{1,2}$ and $f^{(r)}$ for $r > s_{2,1}$.

- If $\sigma = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, then $Y_{1|1}(\sigma) = Y_{1|1}$.

7. Principal W-algebra $W(\pi)$

$\mathfrak{g} = \mathfrak{gl}(l|k)$ is a general linear Lie superalgebra, $(x|y) = \mathrm{str}(xy)$.
Assume that $l \geq k$.

Definition. π is a two-rowed *pyramid:*
k is the number of boxes in the 1-st row,
l is the number of boxes in the 2-nd row.
Each row is a connected horizontal strip.

Definition. The *shift matrix* for π is

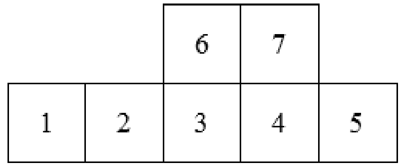
$$\sigma = \begin{pmatrix} 0 & s_{1,2} \\ s_{2,1} & 0 \end{pmatrix}, \text{ where } \pi \text{ has}$$

$s_{2,1}$ columns of hight one on its *left* side and
$s_{1,2}$ columns of hight one on its *right* side,
or if $k = 0$ and $l = s_{2,1} + s_{1,2}$.

- $l = s_{2,1} + k + s_{1,2}$.

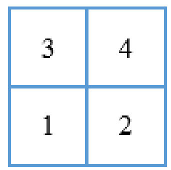
Example.

$\mathfrak{g} = \mathfrak{gl}(5|2)$, $\pi =$



$$\sigma = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$$

$\mathfrak{g} = \mathfrak{gl}(2|2)$, $\pi =$



$$\sigma = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

• Pyramid π defines $\mathbb{Z}$**-grading** on $\mathfrak{g}$:

$$\mathfrak{g} = \oplus_{r \in \mathbb{Z}} \mathfrak{g}(r) \qquad \deg(e_{i,j}) := col(j) - col(i), \qquad \mathfrak{h} := \mathfrak{g}(0)$$

• The explicit **principal (regular)** nilpotent element e is

$$e := \sum_{i,j} e_{i,j} \in \mathfrak{g}_{\bar{0}}$$

summing over all *adjacent pairs* (i, j) of boxes in π.

Example. $\mathfrak{g} = \mathfrak{gl}(5|2)$, $e = e_{1,2} + e_{2,3} + e_{3,4} + e_{4,5} + e_{6,7}$

Remark. $e \in \mathfrak{g}(1)$. We double the degree to agree with the previous definition.

- $\chi(x) := (x|e)$. This is a good $\mathbb{Z}$-grading for χ.

The finite W-algebra $W(\pi)$ associated to the pyramid π is defined as usual.

Remark. In the case when $\mathfrak{g} = \mathfrak{gl}(l|l)$, $\mathfrak{g}^\chi$ is isomorphic to the truncated Lie superalgebra of polynomial currents in $\mathfrak{gl}(1|1)$:

$$\mathfrak{g}^\chi \cong \mathfrak{gl}(1|1) \otimes \mathbb{C}[t]/(t^l)$$

Theorem. *(Brown-Brundan-Goodwin, 2013)*
Assume that e is a principal (regular) nilpotent element.

Special Case: $\mathfrak{g} = \mathfrak{gl}(l|l)$, $\sigma = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Then

$$W(\pi) \cong Y_{1|1}^{l},$$

that is the image of $Y_{1|1}$ under the homomorphism

$$\mathrm{ev}^{\otimes l} \circ \Delta_l : Y_{1|1} \longrightarrow [U(\mathfrak{gl}(1|1))]^{\otimes l},$$

where

$$\Delta_l : Y_{1|1} \longrightarrow Y_{1|1}^{\otimes l},$$

$$\Delta_l := \Delta_{l-1,l} \circ \cdots \circ \Delta_{2,3} \circ \Delta$$

is a homomorphism of associative algebras.

The General Case: $\mathfrak{g} = \mathfrak{gl}(l|k)$.

$$W(\pi) \cong Y_{1|1}^{l}(\sigma) \subset U(\mathfrak{gl}_1)^{\otimes s_{2,1}} \otimes U(\mathfrak{gl}(1|1))^{\otimes k} \otimes U(\mathfrak{gl}_1)^{\otimes s_{1,2}} \cong U(\mathfrak{h})$$

where $U(\mathfrak{gl}_1) := \mathbb{C}[e_{1,1}]$, $l = s_{2,1} + k + s_{1,2}$.

$$Y_{1|1}^{l}(\sigma) \cong Y_{1|1}(\sigma)/I^{l}(\sigma),$$

where $I^{l}(\sigma)$ is the two-sided ideal generated by $d_1^{(r)}$ for $r > k$.

Theorem. *(Y. Peng, J. Algebra, 2015)*
Peng described the finite W-algebra for $\mathfrak{g} = \mathfrak{gl}(M|N)$ associated to a nilpotent e in the case when

the Jordan type of e satisfies the following condition:

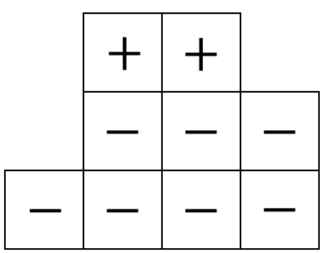
$$e = e_M \oplus e_N,$$

where e_M is principal nilpotent in $\mathfrak{gl}(M|0)$ and

the sizes of the Jordan blocks of e_N are all greater or equal to M.

Signed pyramids: M is the number of boxes with $+$

N is the number of boxes with $-$

$\pi =$ 

The top row of π is the only row assigned with $+$

Example.

$\mathfrak{g} = \mathfrak{gl}(2|7)$, $\pi =$

	$\bar{1}$	$\bar{2}$	
	2	4	6
1	3	5	7

$l = 4$, $\sigma = \left(\begin{array}{c|cc} 0 & 1 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 1 & 0 \end{array}\right)$

Peng proved that $W(\pi) \cong Y^l_{1|n}(\sigma)$, where $n+1$ is the hight of the pyramid π, l is the length of the bottom row, and σ is the shift matrix.

Theorem. *(Y. Peng, Lett. Math. Phys., 2014)*
Let $e \in \mathfrak{gl}(ml|nl)$ be a nilpotent element, whose Jordan blocks are all of size l. Then the associated finite W-algebra is isomorphic to $Y^l_{m|n} = Y_{m|n}/I^l$, where I^l is the 2-sided ideal of $Y_{m|n}$ generated by the elements $\{T^{(r)}_{i,j} | 1 \le i,j \le m+n, r > l\}$. $Y^l_{m|n}$ is identified with the image of $Y_{m|n}$ under the map

$$\mathrm{ev}^{\otimes l} \circ \Delta_l : Y_{m|n} \longrightarrow [U(\mathfrak{gl}(m|n))]^{\otimes l}.$$

8. The queer Lie superalgebra $\mathfrak{g} = \mathbf{Q(n)}$

$$Q(n) = \{ \left(\begin{array}{c|c} A & B \\ \hline B & A \end{array} \right) \mid A, B \text{ are } n \times n \text{ matrices} \}$$

• Supercommutator: $[X, Y] = XY - (-1)^{p(X)p(Y)} YX$.

$e_{i,j}$ and $f_{i,j}$ are standard bases in A and B respectively:

$$e_{i,j} = \left(\begin{array}{c|c} E_{ij} & 0 \\ \hline 0 & E_{ij} \end{array} \right), \quad f_{i,j} = \left(\begin{array}{c|c} 0 & E_{ij} \\ \hline E_{ij} & 0 \end{array} \right)$$

$z = \sum_{i=1}^{n} e_{i,i}$ is a central element

- $Q(n)$ admits an **odd** non-degenerate $\mathfrak{g}$-invariant super-symmetric bilinear form

$$(x|y) := otr(xy) \text{ for } x, y \in \mathfrak{g},$$

$$otr\left(\begin{array}{c|c} A & B \\ \hline B & A \end{array}\right) = trB$$

$$SQ(n) := \{X \in Q(n) \mid \text{otr}X = 0\}.$$

$$\tilde{Q}(n) := SQ(n)/ < z > \text{ is simple for } n \geq 3.$$

9. **Finite W-algebra for $\mathbf{Q(n)}$**

Let $\mathfrak{g} = Q(n)$. Let $\mathfrak{sl}(2) =< e, h, f >$, where

$$e = \sum_{p=1}^{\frac{n}{l}} \sum_{i=1}^{l-1} e_{l(p-1)+i,l(p-1)+i+1},$$

$$f = \sum_{p=1}^{\frac{n}{l}} \sum_{i=1}^{l-1} i(l-i) e_{l(p-1)+i+1,l(p-1)+i},$$

$$h = \sum_{p=1}^{\frac{n}{l}} \sum_{i=1}^{l} (l-2i+1) e_{l(p-1)+i,l(p-1)+i}.$$

Thus e is an even nilpotent element in $Q(n)$.

• Note that e is a nilpotent $n \times n$-matrix, whose Jordan blocks are all of size l.
Jordan blocks is $\frac{n}{l}$.

Example

$$e \in Q(8), \qquad n = 8, \qquad l = 4, \qquad \frac{n}{l} = 2$$

$$e = \left(\begin{array}{cccccccc|cccccccc}
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}\right)$$

- We replace $e = \sum_{p=1}^{\frac{n}{l}} \sum_{i=1}^{l-1} e_{l(p-1)+i,l(p-1)+i+1}$ $(even)$

by

$E = \sum_{p=1}^{\frac{n}{l}} \sum_{i=1}^{l-1} f_{l(p-1)+i,l(p-1)+i+1}$ (odd).

Example

$$e \in Q(8) \qquad (even)$$

$$e = \left(\begin{array}{cccccccc|cccccccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

$$e \in Q(8) \quad (even) \longrightarrow E \in Q(8) \quad (odd)$$

$$E = \left(\begin{array}{cccccccc|cccccccc}
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\hline
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}\right)$$

There is an isomorphism $\mathfrak{g}^* \simeq \Pi(\mathfrak{g})$, where Π is the change of parity.

- Define an even nilpotent $\chi \in \mathfrak{g}^*$ by

$$\chi(x) := (x|E) \text{ for all } x \in \mathfrak{g}$$

Let

$$\mathfrak{g}^E := \{x \in \mathfrak{g} \mid [x, E] = 0\}$$

be the *centralizer* of E in $\mathfrak{g}$. Then

$$\mathfrak{g}^\chi = \mathfrak{g}^E =< \sum_{i=1}^{l-k} e_{l(p-1)+i,l(q-1)+i+k} \mid \sum_{i=1}^{l-k} (-1)^{i+k-1} f_{l(p-1)+i,l(q-1)+i+k} >,$$

where $1 \leq p, q \leq \frac{n}{l}$, $k = 0, 1, \ldots, l-1$.

$$\dim(\mathfrak{g}^E) = (\frac{n^2}{l} | \frac{n^2}{l}).$$

- χ is *regular* nilpotent $\Longleftrightarrow \dim(\mathfrak{g}^\chi) = (n|n) \Longleftrightarrow$ # Jordan blocks in e is one:
$l = n, \frac{n}{l} = 1.$

Example. $\mathfrak{g} = Q(3), \quad l = 3, \quad \dim(\mathfrak{g}^\chi) = (3|3)$

$\mathfrak{g}^\chi$ is spanned by

$$even: \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right), \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right), \left(\begin{array}{ccc|ccc} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

$$odd: \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right), \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right), \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

• $\mathrm{ad}h$ defines an even $\mathbb{Z}$-grading of $\mathfrak{g}$:

$$\mathfrak{g} = \bigoplus_{j=2-2l}^{2l-2} \mathfrak{g}_j,$$

$$\mathfrak{g}_j = \{x \in \mathfrak{g} \mid \mathrm{ad}h(x) = jx\},$$

$$\mathfrak{g}_j = \{0\} \quad \text{for odd } j.$$

• This $\mathbb{Z}$-grading is called Dynkin, and it is good for χ.

$\dim(\mathfrak{g}^\chi) = \dim \mathfrak{g}_0$.

Example

$$\mathfrak{g} = Q(8) : n = 8, \quad l = 4, \quad \frac{n}{l} = 2$$

$$\left(\begin{array}{cccccccc|cccccccc}
0 & 2 & 4 & 6 & 0 & 2 & 4 & 6 & 0 & 2 & 4 & 6 & 0 & 2 & 4 & 6 \\
-2 & 0 & 2 & 4 & -2 & 0 & 2 & 4 & -2 & 0 & 2 & 4 & -2 & 0 & 2 & 4 \\
-4 & -2 & 0 & 2 & -4 & -2 & 0 & 2 & -4 & -2 & 0 & 2 & -4 & -2 & 0 & 2 \\
-6 & -4 & -2 & 0 & -6 & -4 & -2 & 0 & -6 & -4 & -2 & 0 & -6 & -4 & -2 & 0 \\
0 & 2 & 4 & 6 & 0 & 2 & 4 & 6 & 0 & 2 & 4 & 6 & 0 & 2 & 4 & 6 \\
-2 & 0 & 2 & 4 & -2 & 0 & 2 & 4 & -2 & 0 & 2 & 4 & -2 & 0 & 2 & 4 \\
-4 & -2 & 0 & 2 & -4 & -2 & 0 & 2 & -4 & -2 & 0 & 2 & -4 & -2 & 0 & 2 \\
-6 & -4 & -2 & 0 & -6 & -4 & -2 & 0 & -6 & -4 & -2 & 0 & -6 & -4 & -2 & 0 \\
\hline
0 & 2 & 4 & 6 & 0 & 2 & 4 & 6 & 0 & 2 & 4 & 6 & 0 & 2 & 4 & 6 \\
-2 & 0 & 2 & 4 & -2 & 0 & 2 & 4 & -2 & 0 & 2 & 4 & -2 & 0 & 2 & 4 \\
-4 & -2 & 0 & 2 & -4 & -2 & 0 & 2 & -4 & -2 & 0 & 2 & -4 & -2 & 0 & 2 \\
-6 & -4 & -2 & 0 & -6 & -4 & -2 & 0 & -6 & -4 & -2 & 0 & -6 & -4 & -2 & 0 \\
0 & 2 & 4 & 6 & 0 & 2 & 4 & 6 & 0 & 2 & 4 & 6 & 0 & 2 & 4 & 6 \\
-2 & 0 & 2 & 4 & -2 & 0 & 2 & 4 & -2 & 0 & 2 & 4 & -2 & 0 & 2 & 4 \\
-4 & -2 & 0 & 2 & -4 & -2 & 0 & 2 & -4 & -2 & 0 & 2 & -4 & -2 & 0 & 2 \\
-6 & -4 & -2 & 0 & -6 & -4 & -2 & 0 & -6 & -4 & -2 & 0 & -6 & -4 & -2 & 0
\end{array}\right)$$

Let

$$\mathfrak{m} := \bigoplus_{j=1}^{l-1} \mathfrak{g}_{-2j}.$$

The left ideal I_χ and W_χ are defined now as usual.

Let

$$\mathfrak{p} := \bigoplus_{j=0}^{l-1} \mathfrak{g}_{2j}$$

be a parabolic subalgebra of $\mathfrak{g}$ and $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{n}$, where

$$\mathfrak{n} := \bigoplus_{j=1}^{l-1} \mathfrak{g}_{2j}.$$

Since the $\mathbb{Z}$-grading is even, the algebra W_χ can be regarded as a *subalgebra* of $U(\mathfrak{p})$. Let

$$U(\mathfrak{p})^+ := \oplus_{i>0} U(\mathfrak{p})_{2i}.$$

It is a two sided ideal in $U(\mathfrak{p})$ and $U(\mathfrak{p})/U(\mathfrak{p})^+ \cong U(\mathfrak{g}_0)$.

- Let $\vartheta : U(\mathfrak{p}) \longrightarrow U(\mathfrak{g}_0)$ be the natural projection.

Theorem. The restriction to W_χ is the *Harish-Chandra homomorphism*

$$\vartheta : W_\chi \longrightarrow U(\mathfrak{g}_0),$$

which is injective.

10. Generators of W_χ for $Q(n)$

- A. Sergeev recursively defined the elements $e_{i,j}^{(m)}$ and $f_{i,j}^{(m)}$ of $U(Q(n))$:

$$e_{i,j}^{(m)} = \sum_{k=1}^{n} e_{i,k} e_{k,j}^{(m-1)} + (-1)^{m+1} \sum_{k=1}^{n} f_{i,k} f_{k,j}^{(m-1)},$$

$$f_{i,j}^{(m)} = \sum_{k=1}^{n} e_{i,k} f_{k,j}^{(m-1)} + (-1)^{m+1} \sum_{k=1}^{n} f_{i,k} e_{k,j}^{(m-1)},$$

where $e_{i,j}^{(0)} = \delta_{i,j}$ and $f_{i,j}^{(0)} = 0$. *(Lett. Math. Phys. 1983)*

Theorem. $\pi(e_{lp,l(q-1)+1}^{(l+k)})$ and $\pi(f_{lp,l(q-1)+1}^{(l+k)})$ for $p, q = 1, \dots, \frac{n}{l}$ and $k = 0, \dots, l-1$ generate W_χ.

Idea of Proof. Let $P(X)$ be the highest weight component of $Gr_K(X)$. Then

$$P\Big(\pi(e_{lp,l(q-1)+1}^{(l+k)})\Big) = \sum_{i=1}^{l-k} e_{l(p-1)+i,l(q-1)+i+k},$$

$$P\Big(\pi(f_{lp,l(q-1)+1}^{(l+k)})\Big) = \sum_{i=1}^{l-k} (-1)^{i+k-1} f_{l(p-1)+i,l(q-1)+i+k},$$

and these elements form a homogeneous basis of $\mathfrak{g}^\chi$.
$\dim(\mathfrak{g}^\chi) = (\frac{n^2}{l} | \frac{n^2}{l})$.

Corollary.

$$Gr_K W_\chi \simeq S(\mathfrak{g}^\chi)$$

Hence Conjecture is true in this case.

11. Super-Yangian of $Q(n)$

Super-Yangian $Y(Q(n))$ was introduced by M. Nazarov. *(Lecture Notes in Math. 1992)*

• $Y(Q(n))$ is the associative unital superalgebra over $\mathbb{C}$ with the countable set of generators

$$T_{i,j}^{(m)} \text{ where } m = 1, 2, \dots \text{ and } i, j = \pm 1, \pm 2, \dots, \pm n.$$

• The $\mathbb{Z}_2$-grading of the algebra $Y(Q(n))$ is defined as follows:

$$p(T_{i,j}^{(m)}) = p(i) + p(j)$$

where $p(i) = 0$ if $i > 0$ and $p(i) = 1$ if $i < 0$.

• To write down defining relations for these generators we employ the formal series in $Y(Q(n))[[u^{-1}]]$:

$$T_{i,j}(u) = \delta_{i,j} \cdot 1 + T_{i,j}^{(1)} u^{-1} + T_{i,j}^{(2)} u^{-2} + \dots .$$

$$\begin{aligned}
&(u^2 - v^2)[T_{i,j}(u), T_{k,l}(v)] \cdot (-1)^{p(i)p(k)+p(i)p(l)+p(k)p(l)} \\
&= (u + v)(T_{k,j}(u)T_{i,l}(v) - T_{k,j}(v)T_{i,l}(u)) \\
&- (u - v)(T_{-k,j}(u)T_{-i,l}(v) - T_{k,-j}(v)T_{i,-l}(u)) \cdot (-1)^{p(k)+p(l)}
\end{aligned} \tag{1}$$

$$T_{i,j}(-u) = T_{-i,-j}(u) \tag{2}$$

- $Y(Q(n))$ is a *Hopf superalgebra* with *comultiplication* given by

$$\Delta(T_{i,j}^{(r)}) = \sum_{s=0}^{r} \sum_{k} (-1)^{(p(i)+p(k))(p(j)+p(k))} T_{i,k}^{(s)} \otimes T_{k,j}^{(r-s)}.$$

- The *opposite comultiplication* is given by

$$\Delta^{op}(T_{i,j}^{(r)}) = \sum_{s=0}^{r} \sum_{k} T_{k,j}^{(r-s)} \otimes T_{i,k}^{(s)}.$$

Combine the series for $T_{i,j}(u)$ into the single element

$$T(u) = \sum_{i,j} E_{i,j} \otimes T_{i,j}(u)$$

of the algebra $\mathrm{End}(\mathbb{C}^{n|n}) \otimes Y(Q(n))[[u^{-1}]]$.

The element $T(u)$ is invertible and we put

$$T(u)^{-1} = \sum_{i,j} E_{i,j} \otimes \tilde{T}_{i,j}(u).$$

- The assignment $T_{i,j}(u) \mapsto \tilde{T}_{i,j}(u)$ defines the *antipodal map*

$$S : Y(Q(n)) \longrightarrow Y(Q(n)),$$

which is an anti-automorphism of the $\mathbb{Z}_2$-graded algebra $Y(Q(n))$.

Definition. An *anti-homomorphism* $\varphi : \mathbf{A} \to \mathbf{B}$ of associative Lie superalgebras is a linear map, which preserves the $\mathbb{Z}_2$-grading and satisfies for any homogeneous $X, Y \in \mathbf{A}$

$$\varphi(XY) = (-1)^{p(X)p(Y)} \varphi(Y)\varphi(X).$$

Let

$$\Delta_l^{op} : Y(Q(n)) \longrightarrow Y(Q(n))^{\otimes l}$$

where

$$\Delta_l^{op} := \Delta_{l-1,l}^{op} \circ \cdots \circ \Delta_{2,3}^{op} \circ \Delta^{op}$$

- There exists a homomorphism $U : Y(Q(n)) \to U(Q(n))$ defined as follows

$$T_{i,j}^{(r)} \mapsto (-1)^r e_{j,i}^{(r)}, \text{ if } i > 0, j > 0, r > 0,$$
$$T_{i,j}^{(r)} \mapsto (-1)^r f_{j,-i}^{(r)}, \text{ if } i < 0, j > 0, r > 0, T_{i,j}^{(0)} \mapsto \delta_{i,j}.$$

This follows from the results of M. Nazarov and A. Sergeev.

Main Theorem (2017). Let e be an even nilpotent element in $Q(n)$ whose Jordan blocks are each of size l. Then the finite W-algebra for $Q(n)$ is isomorphic to the image of $Y(Q(\frac{n}{l}))$ under the homomorphism

$$U^{\otimes l} \circ \Delta_l^{op} : Y(Q(\frac{n}{l})) \longrightarrow (U(Q(\frac{n}{l})))^{\otimes l}.$$

We proved this theorem in the *regular* case, i.e. $l = n$ in *Adv. Math. 300 (2016)*.

Idea of proof.

• There exists a surjective homomorphism:

$$\varphi : Y(Q(\frac{n}{l})) \longrightarrow W_\chi$$

defined as follows:

$$\varphi(T^{(r)}_{q,p}) = (-1)^r \pi(e^{(l+r-1)}_{lp,l(q-1)+1}), \quad \varphi(T^{(r)}_{-q,p}) = (-1)^r \pi(f^{(l+r-1)}_{lp,l(q-1)+1})$$

for $r = 1, 2, \dots$

In fact, the *Harish-Chandra homomorphism*

$$\vartheta : W_\chi \longrightarrow U(\mathfrak{g}_0),$$

is injective. We have

$$U(\mathfrak{g}_0) \cong U(Q(\frac{n}{l}))^{\otimes l}$$

Then

$$\varphi = \vartheta^{-1} \circ U^{\otimes l} \circ \Delta_l^{op}$$

Hence

$$W_\chi \cong U^{\otimes l} \circ \Delta_l^{op}(Y(Q(\frac{n}{l})))$$

□

Let

$$\Delta_l : Y(Q(n)) \longrightarrow Y(Q(n))^{\otimes l},$$

where

$$\Delta_l := \Delta_{l-1,l} \circ \cdots \circ \Delta_{2,3} \circ \Delta.$$

Definition. The *evaluation homomorphism*

$$ev : Y(Q(n)) \to U(Q(n))$$

is defined as follows

$$T_{i,j}^{(1)} \mapsto -e_{j,i}, \quad T_{-i,j}^{(1)} \mapsto -f_{j,i} \text{ for } i,j > 0, \quad T_{i,j}^{(0)} \mapsto \delta_{i,j}$$

$$T_{i,j}^{(r)} \mapsto 0 \text{ for } r > 1.$$

Theorem.

$$W_\chi \cong ev^{\otimes l} \circ \Delta_l(Y(Q(\frac{n}{l})))$$

Idea of proof. Consider an anti-homomorphism

$$\bar{ev} : Y(Q(n)) \to U(Q(n)),$$

defined by $\bar{ev} := \alpha \circ ev$,
where α is the *principal anti-automorphism* of the enveloping superalgebra $U(\mathfrak{g})$

$$\alpha : X \mapsto -X \quad \text{for all } X \in \mathfrak{g}$$

1) $(U^{\otimes l} \circ \Delta_l^{op})(Y(Q(\frac{n}{l}))) = (\bar{ev}^{\otimes l} \circ \Delta_l)(Y(Q(\frac{n}{l})))$.

Lemma.

$$(\bar{ev} \circ S)(T^{(r)}_{\pm q,p}) = U(T^{(r)}_{\pm q,p}).$$

This implies that

$$(\bar{ev}^{\otimes l} \circ S^{\otimes l} \circ \Delta_l^{op})(T^{(r)}_{\pm q,p}) = (U^{\otimes l} \circ \Delta_l^{op})(T^{(r)}_{\pm q,p}).$$

Finally, the following diagram, where $Y := Y(Q(\frac{n}{l}))$ is commutative:

$$\begin{array}{ccccccc}
Y & \xrightarrow{\Delta} & Y \otimes Y & \xrightarrow{id \circ \Delta} & Y \otimes Y \otimes Y & \xrightarrow{id \circ id \circ \Delta} & \ldots \\
{\scriptstyle S}\big\uparrow & & {\scriptstyle S \otimes S}\big\uparrow & & {\scriptstyle S \otimes S \otimes S}\big\uparrow & & {\scriptstyle S^{\otimes 4}}\big\uparrow \\
Y & \xrightarrow{\Delta^{op}} & Y \otimes Y & \xrightarrow{\Delta^{op} \circ id} & Y \otimes Y \otimes Y & \xrightarrow{\Delta^{op} \circ id \circ id} & \ldots
\end{array}$$

Hence

$$(\bar{ev}^{\otimes l} \circ \Delta_l \circ S)(T^{(r)}_{\pm q,p}) = (U^{\otimes l} \circ \Delta_l^{op})(T^{(r)}_{\pm q,p}).$$

2) $(\bar{ev}^{\otimes l} \circ \Delta_l)(Y(Q(\frac{n}{l}))) = (ev^{\otimes l} \circ \Delta_l)(Y(Q(\frac{n}{l})))$
follows from

$$\bar{ev}^{\otimes l} \circ \Delta_l(T^{(r)}_{\pm q,p}) = (-1)^r ev^{\otimes l} \circ \Delta_l(T^{(r)}_{\pm q,p}).$$

□

Conjecture.

$$W_\chi \cong Y(Q(\frac{n}{l}))/I^l,$$

where I^l is the 2-sided ideal of $Y(Q(\frac{n}{l}))$ generated by the elements

$$\{T^{(r)}_{\pm q,p} \mid 1 \leq q, p \leq \frac{n}{l}, r > l\}$$

Problem: Describe the finite W-algebra for $Q(n)$ associated to an *arbitrary* nilpotent element.

12. W_χ WHEN χ IS REGULAR NILPOTENT

Theorem. *(Adv. Math. 2016)*
If $\mathfrak{g} = Q(n)$ and χ is regular nilpotent, then

(1) the center of W_χ coincides with the center of $U(Q(n))$.

(2) there exist n even and n odd generators in W_χ, such that all even generators commute and generate the polynomial subalgebra of rank n in W_χ, and the commutators of odd generators lie in the center of W_χ.

The proof is based on the surjective homomorphism:

$$\varphi : Y(Q(1)) \longrightarrow W_\chi,$$

and the following relation in $Y(Q(1))$: if $r + s$ is even, then

$$[T_{1,1}^{(r)}, T_{1,1}^{(s)}] = 0.$$

Conjecture. Let $\mathfrak{g}$ be a basic Lie superalgebra and χ be regular nilpotent.

Then it is possible to find a set of generators of W_χ such that even generators commute, and the commutators of odd generators are in the center of $U(\mathfrak{g})$.

- Brown, Brundan and Goodwin proved this Conjecture for $\mathfrak{g} = \mathfrak{gl}(m|n)$.

13. Representations of W_χ when χ is regular nilpotent

Theorem. *(Adv. Math. 2016)*
Let $\mathfrak{g} = Q(n)$ and χ be regular nilpotent. Let M be a simple W_χ-module. Then

$$\dim M \leq 2^{k+1}, \text{ where } k = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

The proof is based on the *Amitsur–Levitzki theorem.*

Theorem. *(A.–L.)* If $A_1, \ldots, A_{2n}$ are $n \times n$ matrices, then

$$\sum_{\sigma \in S_{2n}} sgn(\sigma) A_{\sigma(1)} \ldots A_{\sigma(2n)} = 0.$$

Idea of Proof. The Harish-Chandra homomorphism

$$\vartheta : W_\chi \longrightarrow U(\mathfrak{h})$$

is injective, where

$$\mathfrak{h} := \mathfrak{g}_0 =< e_{i,i} \mid f_{i,i} >, \quad [f_{i,i}, f_{i,i}] = 2e_{i,i}$$

(1) $U(\mathfrak{h})$ satisfies $A.$–$L.$ identity, i.e. for any $u_1, \ldots, u_{2^{k+1}} \in U(\mathfrak{h})$

$$\sum_{\sigma \in S_{2^{k+1}}} sgn(\sigma) u_{\sigma(1)} \ldots u_{\sigma(2^{k+1})} = 0. \tag{$*$}$$

(2) W_χ satisfies $A.$–$L.$ identity, since $W_\chi \cong \vartheta(W_\chi) \subset U(\mathfrak{h})$.

(3) Consider M as a module over the associative algebra W_χ, forgetting the $\mathbb{Z}_2$-grading. Then either M is simple or M is a direct sum of two non-homogeneous simple submodules $M_1 \oplus M_2$.

(a) In the former case $\dim M \leq 2^k$.

Assume $\dim M > 2^k$. Let V be a subspace of dimension $2^k + 1$. By density theorem for any $X_1, \ldots, X_{2^{k+1}} \in \mathrm{End}_{\mathbb{C}}(V)$ one can find $u_1, \ldots, u_{2^{k+1}}$ in W_χ such that $(u_i)_{|V} = X_i$ for all $i = 1, \ldots, 2^{k+1}$. Since $\mathrm{End}_{\mathbb{C}}(V)$ does not satisfy (*) we obtain a contradiction.

(b) In the latter case, we can prove in the same way that $\dim M_1 \leq 2^k$ and $\dim M_2 \leq 2^k$. Therefore $\dim M \leq 2^{k+1}$.

14. Defect

Definition. Let $\mathfrak{g}$ be a basic Lie superalgebra, and let $\triangle$ be the set of roots with respect to a maximal torus in $\mathfrak{g}_{\bar{0}}$. Then the *defect* of $\mathfrak{g}$ is the dimension of a maximal isotropic subspace in the $\mathbb{R}$-span of $\triangle$.

Example.

$\mathrm{def}(\mathfrak{sl}(m|n)) = min(m,n)$,

$\mathrm{def}(\mathfrak{osp}(2m|2n)) = \mathrm{def}(\mathfrak{osp}(2m+1|2n)) = min(m,n)$.

The exceptional Lie superalgebras

$$D(2,1;\alpha), G(3), F(4)$$

have defect one.

Theorem. *(Adv. Math. 2016)*

For a basic Lie superalgebra $\mathfrak{g}$, if χ is regular nilpotent, then all irreducible representations of W_χ are finite-dimensional:

$$\dim M \leq 2^{k+1}$$

$k = d$ **or** $k = d + 1$**, where** d **is the defect of** $\mathfrak{g}$:

- $k = d$, if $\mathfrak{g}$ is of type I: $\mathfrak{g} = \mathfrak{sl}(m|n), \mathfrak{osp}(2|2n)$,

  or $\mathfrak{g}$ is of type II and $\dim(\mathfrak{g}_{\bar{1}}^{\chi})$ is *even:* $\mathfrak{g} = \mathfrak{osp}(2m+1|2n)$ for $m \geq n$,
  $\mathfrak{osp}(2m|2n)$ for $m \leq n$, $\quad G_3$.

- $k = d + 1$, if $\mathfrak{g}$ is of type II and $\dim(\mathfrak{g}_{\bar{1}}^{\chi})$ is *odd:*

  $\mathfrak{g} = \mathfrak{osp}(2m+1|2n)$ for $m < n$, $\quad \mathfrak{osp}(2m|2n)$ for $m > n$, $\quad D(2,1;\alpha)$, $\quad F_4$.

Idea of Proof.

(1) If $\mathfrak{g}$ is of **Type I**, then it admits an **even** good $\mathbb{Z}$-grading for a regular χ.

Then there is an injective homomorphism

$$\vartheta : W_\chi \longrightarrow U(\mathfrak{g}_0).$$

(2) If $\mathfrak{g}$ is of **Type II**, then it admits **no even** good $\mathbb{Z}$-grading for a regular χ.

One can construct an injective homomorphism

$$\vartheta : W_\chi \longrightarrow \bar{W}_\chi^{\mathfrak{s}},$$

where $\bar{W}_\chi^{\mathfrak{s}}$ is "the finite W-algebra" of $\mathfrak{s}$:

$\mathfrak{s}$ is the Levi subalgebra of a parabolic subalgebra $\mathfrak{p}$, such that $\mathfrak{n}^- \subset \mathfrak{m} \subset \mathfrak{p}^-$, where $\mathfrak{n}^-$ is the nilradical of the opposite parabolic $\mathfrak{p}^-$.

$\bar{W}_\chi^{\mathfrak{s}} = (U(\mathfrak{s}) \otimes_{U(\mathfrak{m}^{\mathfrak{s}})} C_\chi)^{\mathfrak{m}^{\mathfrak{s}}}$, where $\mathfrak{m}^{\mathfrak{s}} = \mathfrak{m} \cap \mathfrak{s}$, χ is the restriction of χ on $\mathfrak{s}$.

(3) If χ is regular, then $U(\mathfrak{g}_0)$ (correspondingly, $\bar{W}_\chi^{\mathfrak{s}}$) satisfies the Amitsur–Levitzki identity.

Hence W_χ satisfies the Amitsur–Levitzki identity.

Problem: Classify the finite-dimensional irreducible representations of finite W-algebras.

15. References

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