CLOSURES OF $O(n)$-ORBITS IN THE FLAG VARIETY FOR $GL(n)$

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$$G = GL(n, \mathbb{C}), G/B = \text{ flag variety for } G$$

$$K = O(n, \mathbb{C}), \text{ a symmetric subgroup}$$

We look at closures $\bar{\mathcal{O}}$ of K-orbits in G/B and their singularities, which control much of the infinite-dimensional representation theory of G.

More precisely, we want to understand when an orbit closure $\bar{\mathcal{O}}$ is rationally smooth (with the same relative cohomology as a smooth variety), or smooth.

Start by meeting our K-orbits $\mathcal{O}$: by work of Richardson and Springer, these are parametrized by the set I_n of involutions in the symmetric group S_n. In more detail, we identify G/B with the variety of complete flags $V_0 \subset V_1 \subset \cdots \subset V_n$ in $\mathbb{C}^n$. The group K is the isotropy group for a symmetric nondegenerate bilinear from $(\cdot, \cdot)$ on $\mathbb{C}^n$; a given flag $V_0 \subset \cdots \subset V_n$ lies in the orbit $\overline{\mathcal{O}}_\pi$ corresponding to the involution π if and only if the rank r_{ij} of $(\cdot, \cdot)$ on $V_i \times V_j$ equals the cardinality $\#\{k : 1 \le k \le i, \pi(k) \le j\}$ for all $1 \le i, j \le n$. Thus in particular if $(\cdot, \cdot)$ is nondegenerate on each V_i (the generic case), then $\pi = 1$ and $r_{ij} = \min(i, j)$ for all i, j; the opposite extreme occurs when $V_{\lfloor n/2 \rfloor}$ is totally isotropic and $r_{ii} = 2(i - \lfloor n/2 \rfloor)$ for $i > n/2$; in this case $\pi = w_0$, the longest element of the Weyl group. The standard order relation on orbits (given by containment of their closures) is given by reverse Bruhat order.

Our main tool for characterizing rational smoothness will be a standard tool in combinatorics namely pattern avoidance, but we define this in a nonstandard way: given an involution $\pi = \pi_1 \ldots \pi_n$ in one-line notation, we say that it *includes* the pattern $\mu = \mu_1 \ldots \mu_k$ if there are indices $i_1, \ldots, i_k$ *permuted by* π such that $\pi_{i_j} > \pi_{i_k}$ if and only if $\mu_j > \mu_k$. (The standard definition would omit the condition that the i_j be permuted by μ.) For example, by our definition the involution 65872143 does *not* include the pattern 2143, for even though the indices $2, 1, 4, 3$ occur in that order in π they are not permuted by it. The reason for the difference is that Schubert varieties in type A are parametrized by permutations; for us only the involutions matter. Ultimately this distinction will (conjecturally) make no difference for us; I will say more about this later.

For Schubert varieties there are well-known poset- and graph-theoretic criteria for rational smoothness of $\overline{\mathcal{O}}_\pi$ due to Carrell and Peterson. Both of these refer to the order ideal I_π of involutions lying below π (so above it in the usual Bruhat order). The poset criterion looks at the rank generating function of I_π and asks that it be palindromic as a polynomial. It holds in many settings closely related to ours but not in our setting. The graph criterion does hold in our setting and of course requires that we make I_π into a graph. If n is even, we do this by joining the involutions μ, ν whenever either $\nu = t\mu t \neq \mu$ for some transposition t or $\nu = t\mu$ and $t\mu t = \mu$; in either case we do *not* insist that t be a simple reflection. We say that neighbors $\nu = t\mu t$ of μ are of *type 1*; neighbors $\nu = t\mu$ are of *type 2*. For odd $n = 2m+1$, we define both the graph and the types of neighbors differently: the only neighbors of μ take the form $t\mu t \neq \mu$ for some t. They are said to be of type 1 if the transposition t does not involve the middle index $m+1$ and of type 2 otherwise. In either case (i.e. for any n) $\overline{\mathcal{O}}_\pi$ is rationally smooth only if the degree of w_0, or more generally any conjugate of w_0, is $r(\pi)$, where $r(\pi)$ is the rank function

$$r(\pi) = \lfloor n^2/4 \rfloor - \sum_{i<\pi(i)} (\pi(i) - i - \#\{k : i < k < \pi(i), \pi(k) < i\})$$

To understand this formula, picture the involution π via its *arc diagram*: depict the indices i, lying between 1 and n, as dots in a row, and join the ith dot to the jth one by an arc if the indices i, j are flipped by π. Then the formula for $r(\pi)$ amounts to taking the sum of the lengths of the arcs,subtracting one whenever one arc crosses another, and finally subtracting the result from $\lfloor n^2/4 \rfloor$. In general the degree of any conjugate of w_0 in I_π is at least $r(\pi)$.

Then our main result is

Theorem 1. There is a list of 22 bad patterns such that if π includes any pattern in the list, then some conjugate of w_0 has degree larger than $r(\pi)$. The same holds if π includes the pattern 2143, provided there are an even number of fixed indices between 21 and 43 (e.g. $\pi = 21354687$, but not 2134576.)

The patterns range in length from 4 to 8; the list will be given later. The presence of an extra condition on fixed points for the pattern 2143 is unprecedented in the pattern avoidance literature; by now many modifications of the classical notion of avoidance, motivated by a number of applications, have been considered, but not this one. Moreover, we have

Theorem 2. If π avoids all bad patterns in the list and n is even, then the bottom vertex w_0 has the right degree $r(\pi)$ and the orbit closure $\overline{\mathcal{O}}_\pi$ is rationally smooth. Thus w_0 has degree $r(\pi)$ if and only if $\bar{\mathcal{O}}_\pi$ is rationally smooth.

We conjecture that the first assertion in Theorem 2 holds for all n. For $\bar{\mathcal{O}}_\pi$ to be smooth,

π should avoid the pattern 1324 as well. (Here neither the graph nor the poset criterion applies, but a direct computation of the Jacobian matrix shows that avoiding this pattern is necessary for smoothness.)

To put this result in context, let me discuss orbit closures of different subgroups of G on G/B for which pattern avoidance criteria for rational smoothness are known. Here the orbits are not always parametrized by permutations. For example, if $G - GL(p+q,\mathbb{C}), K = GL(p,\mathbb{C}) \times GL(q,\mathbb{C})$ then K-orbits are parametrized by involutions in S_{p+q} whose fixed points are labelled $+$ or $-$, with the condition that the umber of pairs plus the number of $+$ signs equals p. If we label each such involution by a *clan*, that is, a sequence $(c_1, \ldots, c_{p+q}$ with each c_i either a sign or a natural number, with every natural number occurring either exactly twice or not at all among the c_i, then the bad patterns for rational smoothness are $(1,+,-,1),(1,-,+,1),(1,2,1,2),(1,+,2,2,1),(1,-,2,2,1),(1,2,2,+,1),(1,2,2,-,1),$ $(1,2,2,3,3,1)$; smoothness and rational smoothness are equivalent for such orbit closures. Here the appropriate notion of pattern inclusion pays attention only to which pairs of numbers are equal, not to the sizes of the numbers, so that for example $(3,4,+,-,3,4)$ contains the pattern $(1,2,+,1,2)$. Also, whenever rational smoothness fails, some vertex corresponding to a closed orbit (there is no single bottom vertex in this case) has the wrong degree. For $G = \mathrm{Sp}(2p+2q,\mathbb{C}), K = \mathrm{Sp}(2p,\mathbb{C}) \times \mathrm{Sp}(2q,\mathbb{C}), n = p+q$, orbits are parametrized by clans $(c_1, \ldots, c_{2n})$ that are symmetric in the sense that if c_i is a sign, then c_{2n+1-i} is the same sign; if c_i, c_j are a pair of equal numbers, then $j \neq 2n+1-i$ and c_{2n+1-i}, c_{2n+1-j} are also a pair of equal numbers. Here the bad patterns are the same as in the previous case, *except* that we allow a middle segment $(c_i, \ldots, c_{2n+1-i})$ of a symmetric sequence $c = (c_1, \ldots, c_{2n})$ to parametrize the open orbit for the appropriate symplectic group; the orbit corresponding to c ha rationally smooth closure if the initial and final segments $(c_1, \ldots, c_{i-1}), (c_{2n+2-i}, \ldots, c_{2n})$ avoid the bad patterns, even if c as a whole does not. Once again; this kind of "dispensation" for pattern avoidance has not been seen in the pattern avoidance literature. Smoothness and rational smoothness are again equivalent in this setting. A similar situation (thus again with a dispensation) holds for $G = SO(2n,\mathbb{C}), K = GL(n,\mathbb{C})$.

Now we come to an example much closer to our main one. If $G = GL(2n,\mathbb{C}), K = \mathrm{Sp}(2n,\mathbb{C})$ then orbits $\mathcal{O}_\pi$ are parametrized by *fixed-point-free* involutions in S_{2n}, once again with the reverse Bruhat order. Defining pattern avoidance as above, there is a list of 17 bad patterns such that $\overline{\mathcal{O}}_\pi$ is rationally smooth if and only if π avoids these patterns. The patterns are

351624, 64827153, 57681324, 53281764, 43218765, 65872143, 21654387, 21563487
34127856, 43217856, 34128765, 36154287, 21754836, 63287154, 54821763, 46513287
21768435

and smoothness and rational smoothness are equivalent. Moreover an orbit closure $\bar{\mathcal{O}}_\pi$ is rationally smooth if and only if the bottom vertex w_0 has the right degree $r(\pi)$. I proved this result in 2009, with an assist from Axel Hultman; here the rank symmetry condition on I_π holds whenever $\overline{\mathcal{O}}_\pi$ is rationally smooth. The proof uses a factorization of the rank generating function whenever the bad patterns are avoided, similar to one used by Sara for Schubert varieties of classical type.

Sketch of proof of Theorems 1,2: first show that if π contains a bad pattern, then $\bar{\mathcal{O}}_\pi$ is rationally singular. Easy to check for the patterns π themselves: degree of w_0 is too large in all cases. Considerably trickier than you might expect to extend to involutions including π, since e.g. if n is odd, the vertex w_0 need *not* have the wrong degree. Instead, we check inductively, by adding fixed points and transpositions to π one at a time, that some conjugate of w_0 (so having only one fixed point, if n is odd) always has the wrong degree.

Why does w_0 alone determine rational smoothness if $n = 2m$ is even? To prove this we replace $\overline{\mathcal{O}}_\pi$ by a smaller variety $\mathcal{S}$ whose rational smoothness properties are the same. This is done using Brion's notion of a slice of a variety. Next, again following Brion, we look at a maximal torus T in G and count the dimensions of the subvarieties $V^{T'}$ of an orbit closure $V = \overline{\mathcal{O}}$ fixed by subtori T' of T of codimension one. Each $V^{T'}$ must be rationally smooth and the sum of their dimensions must equal $\dim V$.

Finally, we need to exhibit a punctured neighborhood N of the closed orbit $P = \bar{\mathcal{O}}_{w_0}$ (a single point) in $\mathcal{S}$ that is rationally smooth. We work out explicit determinantal equations realizing the full slice $\mathcal{S}$ as the product of an affine space and a variety with only one singular point (at the origin), whence a suitable punctured neighborhood N is obtained by just removing P from $\mathcal{S}$. This proves the second assertion in Theorem 2. The proof relies heavily on the "extra" edges (corresponding to transpositions of type 2) that appear only for n even, so does not work for odd n.

Now we have to prove that if π avoids all the bad patterns then the vertex w_0 has the right degree $r(\pi)$. Assume first that π is fixed-point-free. We exploit the earlier list of 17 patterns which if avoided by a fixed-point-free involution π guarantees that w_0 has the right number of type 1 neighbors. (The list of bad patterns in Theorem 1 captures the earlier list and the rank functions are the same.)

What about involutions π that are not fixed-point-free? Following a suggestion of Axel Hultman, we first show that that there is a unique smallest fixed-point-free involution $f(\pi)$ above π in the *usual* Bruhat order, for n even. This is constructed by a bumping algorithm similar to RSK. If π is not fixed-point-free, then it must have evenly many fixed points; let these points be $i_1, \ldots, i_{2k}$ in increasing order. For every j between 1 and k, enumerate the pairs of indices flipped by π between i_{2j-1} and i_{2j} *not* encapsulating another such a pair as $(ell_1, \ell_1'), \ldots, .(\ell_m, \ell_m')$. Replace the leftmost fixed point i_{2j-1} in the one-line notation of π by ℓ_1, ℓ_1 by $\ell_2 \ldots, \ell_m$ by i_{2j}, changing the other indices as necessary to make the result an involution. (Thus for example if $\pi = 16573248$, then $f(\pi) = 56781234$.) Then a type 1 neighbor of w_0 lying below π in the reverse Bruhat order is the same as one lying below $f(\pi)$. We know how to count neighbors of this type if $f(\pi)$ avoids all patterns in the list of 17; if $f(\pi)$ includes one of these last patterns, then there is at least one more such neighbor. The number of neighbors so far is typically *less* than $r(\pi)$, as $f(\pi)$ typically has smaller rank than π. Now we have to count the type 2 neighbors lying below π; this is easily read off from the one-line notation of π. In more detail, one looks for the smallest index i such that $\pi(i) \leq i$ and likewise the smallest index j such that $\pi(2m+1-j) \geq 2m+1-j$; the number of type 2 neighbors is then $n+1-\max(i,j)$. Now we can compute the degree of w_0 in I_π. If this is too large, then either π is fixed-point-free already and w_0 has too many type 1 neighbors, or the number of type 2 neighbors picked up by w_0 is larger than the number of type 1 neighbors lost when passing from π to $f(\pi)$. If this occurs, we can "localize" the occurrence, arguing that it must also occur for at most eight indices permuted by π, forcing π to include a bad involution of length at most eight. Such involutions can be checked directly and it can be verified that all contain a bad pattern, whence so does π.

If $n = 2m+1$ is odd, then a similar but more complicated argument shows that for any involution π there is a unique smallest one $f(\pi)$ lying above π in the usual Bruhat order and fixing only the middle index $m+1$; to construct it we first replace π by a higher involution fixing at least $m+1$, then by one fixing only $m+1$. For example, if $\pi = 12435$, then this involution is replaced first by 14325, and then finally by 45312. In this way we show that any involution π avoiding all the bad patterns has w_0 of the right degree $r(\pi)$ in I_π. I believe that a similar argument will show that the conjugates of w_0 in I_π also have the right degree; as mentioned above, I then further conjecture that this forces $\overline{\mathcal{O}}_\pi$ to be rationally smooth.

Our list of bad patterns, in order of increasing length:

2143

14325, 21543, 32154

154326, 124356, 351624, 132546, 426153 153624, 351426, 216543, 432165

5271643, 5472163, 1657324, 4651327

57681324, 65872143, 34127856, 64827153, 13247856, 34125768

Notice that our first (half)-bad pattern 2143, together with the bad one 1324 for smoothness, are exactly the reverses of the bad patterns $3412, 4231$ for rational smoothness of Schubert varieties in type A (Lakshmibai-Sandhya); they are reversed because the order for containment of closures is the reverse of the Bruhat order. Finally, one last conjecture: an involution π contains one of the above patterns in my sense if and only if it does so in the usual sense (omitting the condition that the indices in the bad pattern be permuted by the involution π.)