Classification of module spectra and Franke's algebraicity conjecture

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Want to understand stable homotopy groups of spheres

$\underbrace{\pi_k \mathbb{S} := \operatorname{colim}_{n\to\infty} \pi_{k+n} S^n}$, $\pi_e X = [S^e, X]$

k	0	1	2	3	4	5	6	7
π_k	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/240$
	deg	Hopf		Hopf				Hopf

$\boxed{\pi_* \mathbb{S}}$ - graded ring $\pi_* \mathbb{S} = \bigoplus_{k\geq 0} \pi_k \mathbb{S}$

$\pi_{>0} \mathbb{S}$ - nilpotent elements (Nishida)

$\pi_k \mathbb{S}$ - finite $k>0$ (Serre)

$\pi_k \mathbb{S} \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & k=0 \\ 0 & \text{else} \end{cases}$

cohomology theories:

$E^n : \mathrm{Top}_*^{op} \longrightarrow Ab$, $n\in\mathbb{Z}$

axioms (LES, MV, ...)

<u>Examples.</u> - $H^n(-, A)$ - sing. coh

- $KU^n(-)$, $KO^n(-)$, K-theory
- Cobordism $MO^n(-)$, $MSO^n(-)$, cx cobordism $MU^n(-)$

Adams-Novikov spectral sequence

Nilpotence (Devinatz-Hopkins-Smith)

$\alpha \in \pi_* \mathbb{S}$ α is nilpotent $\Leftrightarrow$ $MU^*(\alpha)$ is zero.

$\alpha : \Sigma^? X \to X$

finite complex

Brown: Such cohomology theories are representable

$E_0, E_1, E_2, \ldots$

$E^n(X) \cong [X, E_n]$

$\tilde{H}^n(X, A) \cong [X, K(A,n)]$

EM space

$E^n(X) \cong E^{n+1}(\Sigma X)$

$\Updownarrow$

$E_n \longrightarrow \Omega E_{n+1} \qquad n \geq 0$

$\Updownarrow$

$\Sigma E_n \longrightarrow E_{n+1}$

<u>Def.</u> A spectrum is a sequence of spaces $\underline{E} = \{E_n\}_{n \geq 0}$ together with $\Sigma E_n \longrightarrow E_{n+1}$

$\boxed{\pi_k E} = \operatorname{colim}_{n \to \infty} \pi_{k+n} E_n$

$\underbrace{\mathbb{S} = \{S^n\}_{n \geq 0} \qquad \Sigma S^n \xrightarrow{\cong} S^{n+1}}$

$\pi_k \mathbb{S}$

∞-category of spectra by Sp.

Sp "$\iff$" cohomology theory

monoids in Sp "$\iff$" multiplicative coh. theories

$E \otimes_{\mathbb{S}} E \to E$

$\underbrace{(Sp, \otimes_{\mathbb{S}}, \mathbb{S})}$ sym monoidal stable, ∞-category

<u>Def.</u> A monoid in $(Sp, \otimes, \mathbb{S})$ is called a ring spectrum. (ring spectrum = $= A_\infty$-ring spectrum $=$ E_1-ring)

<u>Examples.</u> $\boxed{\mathbb{S}}$, $\boxed{\pi_* \mathbb{S}}$

$\underline{E}$ ring spectrum $\quad \pi_* E = \bigoplus_{k \in \mathbb{Z}} \pi_k E$ - graded ring

$\overline{KU}$, KO - represent K-theories;

$\overline{MU}$ - complex cobordism

A ring $\Rightarrow$ $\overline{HA}$ corresponding EM ring spectrum representing singular cohomology

$M \in E\text{-Mod}$ = modules over E

$M \in Sp$, $E \otimes_{\mathbb{S}} M \longrightarrow M$. + ...

X $\boxed{\mathbb{S}\text{-Mod}} = Sp$ - difficult (not algebraic Schwede)

$Ho(E\text{-Mod})$ - same objects as E-Mod morphisms homotopy classes

$\uparrow$ try to find an algebraic model

$\checkmark Ho(Sp_{\mathbb{Q}}) = Ho(\mathbb{S}_{\mathbb{Q}}\text{-Mod}) \sim \underbrace{\text{graded } \mathbb{Q}\text{-vector spaces}}$

$\pi_* \mathbb{S} \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & k = 0 \\ 0 & \text{else} \end{cases}$

? MU-Mod ? difficult

MU-Mod $\pi_* MU \cong \mathbb{Z}[x_1, x_2, x_3, \dots]$
p prime $|x_i| = 2i$

$MU_{(p)} \simeq \bigvee \Sigma^? BP$ BP ring spectrum

$\pi_* BP \cong \mathbb{Z}[v_1, v_2, v_3, \dots]$ $|v_i| = 2(p^i - 1)$

? BP-Mod ? difficult Brown-Peterson

$BP\langle n \rangle$, $\pi_* BP\langle n\rangle \cong \mathbb{Z}[v_1, v_2, \dots, v_n]$

$E(n) = BP\langle n\rangle[v_n^{-1}]$ Johnson-Wilson
$\pi_* E(n) \cong \mathbb{Z}[v_1, v_2, \dots, v_{n-1}, v_n^{\pm 1}]$

$K(n)$ - Morava K-theory $E(n)/(p, v_1, \dots, v_{n-1})$
$\pi_* K(n) \cong \mathbb{F}_p[v_n^{\pm 1}]$ $|v_i| = 2(p^i - 1)$

$\boxed{K(n)\text{-Mod}}$, $BP\langle n\rangle$-Mod $E(n)$-Mod, ✓

A_* - graded ring (i.e. $\pi_* E$)

$\mathcal{D}(A_*)$ = differential graded modules over A_* ∞ category

$A_* \otimes M_* \to M_*$, $d: M_n \to M_{n-1}$
$d^2 = 0$, A_*-linear

Folklore: $Ho(K(n)\text{-Mod}) \simeq Ho(\mathcal{D}(\pi_* K(n))) \simeq$ graded modules over $\mathbb{F}_p[v_n^{\pm 1}]$ ✓

$Ext^i_{\pi_* K(n)} = 0$, $i > 0$

Spectral sequence
$$Ext^{t,s}_{\pi_* E}(\pi_* X, \pi_* Y) \Rightarrow [X, Y]_*^{E\text{-Mod}}$$
$$\checkmark\ [X, Y]^{K(n)\text{-Mod}} \cong Hom_{\pi_* K(n)}(\pi_* X, \pi_* Y)$$

Example. KU, $\pi_* KU \cong \mathbb{Z}[u^{\pm 1}]$, 2 sparse,
$|u| = 2$, u-Bott class

$$Ho(KU\text{-Mod}) \simeq Ho(\mathcal{D}(\pi_* KU)) \quad \checkmark$$

UCT
$$0 \to Ext^1_{\pi_* KU}(\pi_* X, \pi_* Y[1]) \to [X, Y]^{KU} \to Hom_{\pi_* KU}(\pi_* X, \pi_* Y) \to 0$$

homological (projective) dim $\pi_* KU = 1 < 2$,
dim, $\max\{i \mid Ext^i \neq 0\}$

(... Wolbert, Greenlees)

(Bousfield, Wolbert, Greenlees)

Def. A_* is q-sparse if $A_n = 0$ unless $n \equiv 0 \mod q$.

Theorem (P.–Pstrągowski) (Franke's conjecture for modules) Let R be a E_1-ring (ring spectrum) such that $\pi_* R$ is q-sparse $(q \geq 1)$ and suppose

$$d = \dim \pi_* R < q.$$

Then $h_{q-d}(R\text{-Mod}) \sim h_{q-d}(\underset{\substack{\uparrow \\ \text{algebraic}}}{\mathcal{D}}(\pi_* R))$.

Rmk. $\mathcal{C}$, $h_n\mathcal{C}$ same objects as $\mathcal{C}$

$$\mathrm{Map}_{h_n\mathcal{C}}(X,Y) = \tau_{\leq n}\mathrm{Map}_{\mathcal{C}}(X,Y)$$

$$h_1\mathcal{C} = \mathrm{Ho}(\mathcal{C})$$

Examples. $R = K(n)$ $\quad \pi_* R \cong \mathbb{F}_p[v_n^{\pm 1}]$

$|v_n| = 2(p^n-1) = q$ -sparse

$\dim \pi_* R = 0 < 2(p^n - 1)$

$$h_{2(p^n-1)}(K(n)\text{-Mod}) \sim h_{2(p^n-1)}(\mathcal{D}(\mathbb{F}_p[v_n^{\pm1}]))$$

$$h_{2(p^n-1)+1}(K(n)\text{-Mod}) \not\sim h_{2(p^n-1)+1}\mathcal{D}(\mathbb{F}_p[v_n^{\pm1}])$$

- $\tau_{\leq 2(p^n-1)} k(n)$, not Eilenberg–MacLane $(Q_n \in \mathcal{A}_p^*)$

$K(n)$, $E(n)$, MU, ... $\quad \Omega^\infty K(n) \not\simeq EM$

- $BP\langle n\rangle$ $\quad n+1 < 2(p-1)$
- $E(n)$ $\quad n < 2(p-1)$
- $ko_{(p)}$ $\quad d = 2 < 4 = q$ $\quad$ p odd

Idea of the proof: Obstruction theory

Goerss–Hopkins, Baues, Blanc, Dwyer, ...

$$\underbrace{R\text{-Mod} = \mathcal{M}_\infty \dashrightarrow \mathcal{M}_e}_{} \to \mathcal{M}_{e-1} \to \cdots \to \mathcal{M}_0 = \pi_* R\text{-Mod}$$

e integer

$$X \in \mathcal{M}_e \subseteq \mathrm{Fun}^\times\left(((R\text{-Mod})^{\text{fin, proj}})^{op}, \mathrm{Spaces}_*\right)$$

X ℓ-truncated $\qquad X^{\circ}[-1] \xrightarrow[(\ell-1)\text{-equivalence}]{} JLX$

$\mathcal{M}_\ell \rightsquigarrow \mathcal{M}_{\ell+1}$ $\underbrace{Ext^{\leq \ell+3}}$

$\underbrace{\text{General result}}$

Thm (P.–Pstragowski, Franke's alg. conjecture 96)
Let $\mathcal{A}$ be a Grothendieck abelian category
$d = \dim_{inj} \mathcal{A} < \infty$. $[1] : \mathcal{A} \xrightarrow{\sim} \mathcal{A}$,
$\mathcal{B} \subseteq \mathcal{A}$ - Serre subcategory for some q

$$\mathcal{A} \sim \mathcal{B} \oplus \mathcal{B}[1] \oplus \cdots \oplus \mathcal{B}[q-1]$$

$d < q$. $\underbrace{\mathcal{C}}$ stable ∞-category presentable

Suppose we have $\underbrace{h}: Ho(\mathcal{C}) \longrightarrow \mathcal{A}$
homology theory $h(\Sigma X) \cong h(X)[1]$
for any injective $I \in \mathcal{A}$, $\exists \tilde{I} \in \mathcal{C}$
$h(\tilde{I}) = I$

$$[X, I] \xrightarrow[h]{\cong} Hom_{\mathcal{A}}(h(X), \tilde{I}).$$

Then $h_{q-d}\mathcal{C} \sim h_{q-d} \mathcal{D}^{[1]}(\mathcal{A})$
$\qquad\qquad\qquad\qquad\qquad\qquad \ni$
Ex. E_n-local spectra $\qquad C \in \mathcal{A}$
$\qquad\qquad\qquad\qquad\qquad\qquad C \xrightarrow{d} C[1] \quad d^2 = 0$

$\underbrace{2p-2 > n^2+n}$

(Pstragowski : $p > n^2 + n + 1$)

$Ho(\mathcal{D}(\mathbb{Q})) \sim$ graded $\mathbb{Q}$-VS