Moment categories and operads

Clemens Berger

University of Nice-Sophia Antipolis

University of Haifa
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Summary (active/inert factorisation system)

$\overset{moments}{\rightsquigarrow}$ moment category $\overset{units}{\rightsquigarrow}$ operad-type $\overset{plus}{\rightsquigarrow}$ Segal presheaf

$\mathbb{C}$	$\mathbb{C}$-operad	$\mathbb{C}$-monoid	$\mathbb{C}_\infty$-monoid
Γ	sym. operad	comm. monoid	E_∞-space
Δ	non-sym. operad	assoc. monoid	A_∞-space
Θ_n	n-operad	n-monoid	E_n-space
Ω	tree-hyperoperad	sym. operad	∞-operad
$\Gamma_\updownarrow$	graph-hyperoperad	properad	∞-properad

Related concepts (replacing "inert part" with $\rightsquigarrow$)

Operator category (Barwick $\rightsquigarrow$ pullback structure)
Operadic category (Batanin-Markl $\rightsquigarrow$ fibre structure)
Feynman category (Kaufmann-Ward $\rightsquigarrow$ sym. monoidal structure)
Categorical pattern (Chu-Haugseng $\rightsquigarrow$ ∞-categorical context)

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Definition (moment category)

A *moment category* is a category $\mathbb{C}$ with an *active/inert* factorisation system $(\mathbb{C}_{act}, \mathbb{C}_{in})$ such that

(1) each inert map admits a unique active retraction;

(2) if the left square below commutes then the right square as well

$$
\begin{tikzcd}
A \arrow[r, "f", "\shortmid"{marking}] & B \\
A' \arrow[u, tail, "i"] \arrow[r, "\shortmid"{marking}, "g"'] & B' \arrow[u, tail, "i'"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f", "\shortmid"{marking}] \arrow[d, "r"', "\shortmid"{marking}] & B \arrow[d, "r'", "\shortmid"{marking}] \\
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where r, r' are the active retractions of i, i' provided by (1).

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Lemma (inert subobjects vs moments)

For each object A of a moment category $\mathbb{C}$ there is a bijection between *inert subobjects* of A and *moments* of A, i.e. endomorphisms $\phi : A \to A$ sth. $\phi = \phi_{in}\phi_{act} \implies \phi_{act}\phi_{in} = 1_A$.

Put $m_A = \{\phi \in \mathbb{C}(A, A) \,|\, \phi_{act}\phi_{in} = 1_A\}$
For $f : A \to B$ define $f_* : m_A \to m_B$ by

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, shift right, "\phi_{act}"'] & B \arrow[d, shift left, "\psi_{act}"] \\
A_\phi \arrow[u, shift right, "\phi_{in}"'] \arrow[r, "f'"'] & B_\psi \arrow[u, shift left, "\psi_{in}"]
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Proposition (left regular band – skew-commutativity)

The moment set m_A is a submonoid of $\mathbb{C}(A, A)$ consisting of idempotent elements satisfying the relation $\phi\psi\phi = \phi\psi$.

Example (Segal's category $\Gamma \rightsquigarrow \Gamma^{\mathrm{op}}$ = finite sets and partial maps)

- $\underline{m} \xrightarrow{(\underline{n}_1,\dots,\underline{n}_m)} \underline{n}$ active provided $\underline{n}_1 \cup \cdots \cup \underline{n}_m = \underline{n}$. (partition)
- $\underline{m} \xrightarrow{(\underline{n}_1,\dots,\underline{n}_m)} \underline{n}$ inert provided all $\underline{n}_i$ are singleton. (embedding)

Example (simplex category Δ)

- $[m] \xrightarrow{f} [n]$ is active provided f is endpoint-preserving, i.e. $f(0) = 0, f(m) = n$.
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Definition (units, elementary moments, nilobjects)

- A moment ϕ is *centric* if ϕ_{in} is the only inert section of ϕ_{act}.
- A *unit* is an object U sth. 1_U is the only centric moment but $m_U \neq \{1_U\}$, and every active map with target U admits exactly one inert section.
- A moment is *elementary* if it splits over a unit. The set of elementary moments of A is denoted $\mathrm{el}_A \subset m_A$.
- An object without elementary moments is called a *nilobject*.

Example (Γ and Δ)

- $\underline{0}$ is the nilobject, and $\underline{1}$ the unit of Γ. Elementary inert subobjects $\underline{1} \rightarrowtail \underline{n}$ are elements. Cardinality of $\mathrm{el}_{\underline{n}}$ is n.
- $[0]$ is the nilobject, and $[1]$ the unit of Δ. Elementary inert subobjects $[1] \rightarrowtail [n]$ are segments. Cardinality of $\mathrm{el}_{[n]}$ is n.

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Definition ($\mathbb{C}$-operads for unital moment categories $\mathbb{C}$)

A $\mathbb{C}$-operad $\mathcal{O}$ in a symmetric monoidal category $(\mathbb{E}, \otimes, I_\mathbb{E})$ assigns to each object A of $\mathbb{C}$ an object $\mathcal{O}(A)$ of $\mathbb{E}$, together with

- a unit $I_\mathbb{E} \to \mathcal{O}(U)$ in $\mathbb{E}$ for each unit U of $\mathbb{C}$;
- a unital, associative and equivariant composition $\mathcal{O}(A) \otimes \mathcal{O}(f) \to \mathcal{O}(B)$ for each active $f : A \longrightarrow\!\!\!\!\!\!\!+\;\; B$, where $\mathcal{O}(f) = \otimes_{\alpha \in \mathrm{el}_A} \mathcal{O}(B_{f_*(\alpha)})$.

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- Γ-operads=symmetric operads: $\mathcal{O}_m \otimes \mathcal{O}_{n_1} \otimes \cdots \otimes \mathcal{O}_{n_m} \to \mathcal{O}_{n_1+\cdots+n_m}$ for each $\underline{m} \longrightarrow\!\!\!\!\!\!\!+\;\; \underline{n}$.
- Δ-operads=non-symmetric operads: $\mathcal{O}_m \otimes \mathcal{O}_{n_1} \otimes \cdots \otimes \mathcal{O}_{n_m} \to \mathcal{O}_{n_1+\cdots+n_m}$ for each $[m] \longrightarrow\!\!\!\!\!\!\!+\;\; [n]$.

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Remark (moment category structure on Θ_n)

- Objects of Θ_n correspond to n-level trees.
- There is a unique *unit* U_n, the linear tree of height n.
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- Active maps $S \twoheadrightarrow T$ correspond to Batanin's S_*-indexed decompositions of T_*, where T_* is the *n-graph* defined by the *inert subobjects* of T whose domains are subobjects of U_n.

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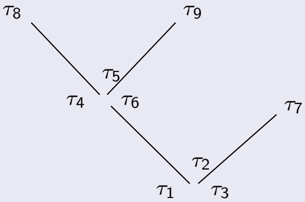

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Definition (hypermoment category)

A *hypermoment category* $\mathbb{C}$ comes equipped with an active/inert factorisation system and $\gamma_{\mathbb{C}} : \mathbb{C} \to \Gamma$ such that

- $\gamma_{\mathbb{C}}$ preserves active (resp. inert) morphisms;
- for each A and $\underline{1} \rightarrowtail \gamma_{\mathbb{C}}(A)$, there is an ess. unique inert lift $U \rightarrowtail A$ in $\mathbb{C}$ such that U satisfies the second unit-axiom.

Example (dendroidal category Ω of Moerdijk-Weiss)

- objects (dendrices) are finite rooted trees with leaves.
- every morphism decomposes into a degeneracy followed by active mono followed by inert mono.
- active mono = inner face = dendrix insertion
  inert mono = outer face = dendrix embedding
- $\gamma_\Omega : \Omega \to \Gamma$ takes a dendrix to its vertex set.
- units = corollas C_n, one for each $n \in \mathbb{N}$.

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A *hypermoment category* $\mathbb{C}$ comes equipped with an active/inert factorisation system and $\gamma_{\mathbb{C}} : \mathbb{C} \to \Gamma$ such that

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Example (dendroidal category Ω of Moerdijk-Weiss)

- objects (dendrices) are finite rooted trees with leaves.
- every morphism decomposes into a degeneracy followed by active mono followed by inert mono.
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Definition (plus construction for unital hypermoment categories $\mathbb{C}$)

- A $\mathbb{C}$-*tree* $([m], A_0 \relbar\joinrel+\joinrel\rightarrow \cdots \relbar\joinrel+\joinrel\rightarrow A_m)$ consists of $[m]$ in Δ and a functor $A_\bullet : [m] \to \mathbb{C}_{act}$ such that A_0 is a unit in $\mathbb{C}$.
- A $\mathbb{C}$-*tree morphism* (ϕ, f) consists of $\phi : [m] \to [n]$ and a nat. transf. $f : A \to B\phi$ sth. $f_i : A_i \to B_{\phi(i)}$ is inert for $i \in [m]$.
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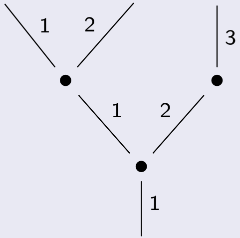

$$
\begin{tikzcd}
\underline{3} \\
\underline{2} \arrow[u, "(\{1{,}2\}{,}\{3\})"'] \\
\underline{1} \arrow[u, "(\{1{,}2\})"']
\end{tikzcd}
$$

Remark (reduced dendrices)

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\begin{tikzcd}
\Gamma^+_{mono} \arrow[r, hook, "="] \arrow[d, hook] & \Omega_{open,pruned} \arrow[d, hook] \\
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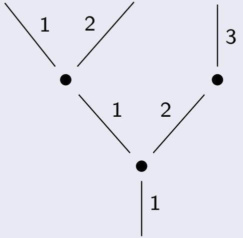

$$\begin{array}{c}\underline{3}\\ \uparrow(\{1,2\},\{3\})\\ \underline{2}\\ \uparrow(\{1,2\})\\ \underline{1}\end{array}$$

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Definition (extensionality)

A hypermoment category $\mathbb{C}$ is *extensional* if pushouts of inert maps along active maps exist, are inert and preserved by $\gamma_{\mathbb{C}}$.

Proposition ($\mathbb{C}$-tree insertion for extensional $\mathbb{C}$)

$\mathbb{C}$-trees can be inserted into vertices of $\mathbb{C}$-trees. There exists a Feynman category $\mathcal{F}_{\mathbb{C}}$ such that ($\mathbb{C}$-operads)$\simeq$($\mathcal{F}_{\mathbb{C}}$-algebras).

Theorem (monadicity for extensional $\mathbb{C}$)

The forgetful functor from $\mathbb{C}$-operads to $\mathbb{C}$-collections is monadic.

Remark

$\mathcal{F}_{\Gamma}$ is the coloured symmetric operad of finite rooted trees whose algebras are symmetric operads.

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Definition (Segal core for strongly unital $\mathbb{C}$)

The *Segal core* $\mathbb{C}_{\mathrm{Seg}}$ is the subcategory of $\mathbb{C}_{in}$ spanned by nil- and unit-objects. $\mathbb{C}$ is *strongly unital* if $\mathbb{C}_{\mathrm{Seg}}$ is dense in $\mathbb{C}_{in}$.

$\mathbb{C}$	Δ	Θ_n	Ω	$\Gamma_{\updownarrow}$
$\mathbb{C}_{\mathrm{Seg}}$	$[0] \rightrightarrows [1]$	cell-incl. of glob. n-cell	edge-incl. of corollas	edge-incl. of dir. corollas
$\mathbb{C}$-gph	graph	n-graph	multigraph	dir. multigraph
$\mathbb{C}$-cat	category	n-category	col. operad	col. properad

Theorem (coloured monadicity for strongly unital $\mathbb{C}$)

The forgetful functor from $\mathbb{C}$-categories to $\mathbb{C}$-graphs is monadic.

Thanks for your attention !

Definition (Segal core for strongly unital $\mathbb{C}$)

The *Segal core* $\mathbb{C}_{\mathrm{Seg}}$ is the subcategory of $\mathbb{C}_{in}$ spanned by nil- and unit-objects. $\mathbb{C}$ is *strongly unital* if $\mathbb{C}_{\mathrm{Seg}}$ is dense in $\mathbb{C}_{in}$.

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$\mathbb{C}$-gph	graph	n-graph	multigraph	dir. multigraph
$\mathbb{C}$-cat	category	n-category	col. operad	col. properad

Theorem (coloured monadicity for strongly unital $\mathbb{C}$)

The forgetful functor from $\mathbb{C}$-categories to $\mathbb{C}$-graphs is monadic.

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