About polynomiality of the Poisson semicentre for parabolic subalgebras

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Aim.

Let $\mathfrak{a}$ be a complex algebraic Lie algebra and A be its adjoint group.
Denote by $S(\mathfrak{a})$ the symmetric algebra of $\mathfrak{a}$.
We want to study the polynomiality of the algebra of invariants $S(\mathfrak{a})^A = S(\mathfrak{a})^{\mathfrak{a}} =: Y(\mathfrak{a})$, especially when $\mathfrak{a}$ is a (truncated) parabolic subalgebra.

Results where polynomiality holds (the list is not exhaustive !).

- When $\mathfrak{a}$ is semisimple, then $Y(\mathfrak{a})$ is polynomial in $\operatorname{rank}\mathfrak{a}$ homogeneous generators (Chevalley, 1968).
- When $\mathfrak{a} = \mathfrak{p}_\Lambda$ is the canonical truncation of a (bi)-parabolic subalgebra of $\mathfrak{g}$ simple of type A or C, then $Y(\mathfrak{a})$ is polynomial in $\operatorname{ind}\mathfrak{p}_\Lambda \leq \operatorname{rank}\mathfrak{g}$ homogeneous generators (Joseph - F-M, 2005).
- When $\mathfrak{a} = \mathfrak{g}^x$ (centralizer of $x \in \mathfrak{g}$), with $\mathfrak{g}$ simple of type A or C and x is any nilpotent element of $\mathfrak{g}$ or when $\mathfrak{g}$ is simple (outside type E_8) and x is the highest root vector of $\mathfrak{g}$, then $Y(\mathfrak{a})$ is polynomial in $\operatorname{rank}\mathfrak{g}$ homogeneous generators (Panyushev - Premet - Yakimova, 2007).

Where polynomiality holds.

- When $\mathfrak{a} = \mathfrak{p} \ltimes (\mathfrak{g}/\mathfrak{p})^a$ (parabolic contraction) and $\mathfrak{p}$ a parabolic subalgebra of $\mathfrak{g}$ simple of type A or C, or when $\mathfrak{p} = \mathfrak{b}$ the Borel subalgebra of $\mathfrak{g}$ simple of any type, then $Y(\mathfrak{a})$ is polynomial in $\mathrm{rank}\,\mathfrak{g}$ homogeneous generators (Panyushev - Yakimova, 2012, 2013).

Counter-examples.

- When $\mathfrak{a} = \mathfrak{p}_\Lambda = \mathfrak{g}^x$ is the centralizer of the highest root vector in $\mathfrak{g}$ simple of type E_8, then $Y(\mathfrak{a})$ is not polynomial (Yakimova, 2007).
- When $\mathfrak{a} = \mathfrak{g}^x$ is the centralizer of some $x \in \mathfrak{g}$ with $\mathfrak{g}$ simple of type D_7, then $Y(\mathfrak{a})$ is not polynomial (Charbonnel - Moreau, 2016).

The case of parabolic subalgebras.

From now on, we focus on a parabolic subalgebra $\mathfrak{p}$ *of a complex simple Lie algebra* $\mathfrak{g}$.
Set, for $\lambda \in \mathfrak{p}^*$, $S(\mathfrak{p})_\lambda := \{s \in S(\mathfrak{p}) \mid \forall x \in \mathfrak{p},\ (\operatorname{ad} x)(s) = \lambda(x)\, s\}$.
Then the Poisson semicentre of $S(\mathfrak{p})$ is

$$Sy(\mathfrak{p}) = \bigoplus_{\lambda \in \mathfrak{p}^*} S(\mathfrak{p})_\lambda,$$

while the Poisson centre is $Y(\mathfrak{p}) = S(\mathfrak{p})_0$.
We have that $Sy(\mathfrak{p}) = S(\mathfrak{p})^{\mathfrak{p}'}$ where $\mathfrak{p}' = [\mathfrak{p},\, \mathfrak{p}]$ and $Y(\mathfrak{p}) \subset Sy(\mathfrak{p})$.
If $\mathfrak{p} = \mathfrak{g}$, then $Y(\mathfrak{g}) = Sy(\mathfrak{g})$.
Otherwise $Y(\mathfrak{p}) = \mathbb{C} \subsetneq Sy(\mathfrak{p})$.
Denote by $\mathfrak{p}_\Lambda$ the canonical truncation of $\mathfrak{p}$, that is, the largest subalgebra of $\mathfrak{p}$ which vanishes on each weight of $Sy(\mathfrak{p})$. Then $Sy(\mathfrak{p}_\Lambda) = Y(\mathfrak{p}_\Lambda) = Sy(\mathfrak{p})$.

Notations.

Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ simple, and a set π of simple roots.
The set of roots of $\mathfrak{g}$ is denoted by $\Delta = \Delta^+ \sqcup \Delta^-$ and the root subspace of $\mathfrak{g}$ corresponding to $\alpha \in \Delta$ is denoted by $\mathfrak{g}_\alpha$.
Then $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$, where $\mathfrak{n}^-$, resp. $\mathfrak{n}$, is the largest nilpotent subalgebra of $\mathfrak{g}$ corresponding to Δ^-, resp. to Δ^+.
Fix a subset $\pi' \subset \pi$ and denote by $\Delta^+_{\pi'}$ the subset of Δ^+ generated by π'.
Let $\mathfrak{p} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}_{\pi'}$ be a parabolic subalgebra of $\mathfrak{g}$ where $\mathfrak{n}_{\pi'}$ is the largest nilpotent subalgebra of $\mathfrak{g}$ corresponding to $\Delta^+_{\pi'}$.
Then the canonical truncation $\mathfrak{p}_\Lambda$ of $\mathfrak{p}$ verifies $\mathfrak{p}_\Lambda = \mathfrak{n}^- \oplus \mathfrak{h}_\Lambda \oplus \mathfrak{n}_{\pi'}$ with $\mathfrak{h}_\Lambda \subset \mathfrak{h}$.

Lower and upper bounds for $Sy(\mathfrak{p})$.

For any $\mathfrak{h}$-module $M = \bigoplus_{\nu\in\mathfrak{h}^*} M_\nu$ with finite dimensional weight subspaces M_ν, we define its formal character

$$\mathrm{ch}\, M = \sum_{\nu\in\mathfrak{h}^*} \dim M_\nu e^\nu.$$

Theorem 1 (Joseph - F-M, 2005)

There are two polynomial algebras $\mathscr{A}$ and $\mathscr{B}$, which are also $\mathfrak{h}$-modules, such that

$$\mathrm{ch}\,\mathscr{A} \leq \mathrm{ch}\, Sy(\mathfrak{p}) \leq \mathrm{ch}\,\mathscr{B}.$$

Moreover if $\mathrm{ch}\,\mathscr{A} = \mathrm{ch}\,\mathscr{B}$, then $Sy(\mathfrak{p})$ is a polynomial algebra.
It is in particular the case when $\mathfrak{g}$ is simple of type A or C.

Criterion

There is a combinatorial criterion, which establishes when $\mathrm{ch}\,\mathscr{A} = \mathrm{ch}\,\mathscr{B}$, so when $Sy(\mathfrak{p}) = Y(\mathfrak{p}_\Lambda)$ is a polynomial algebra.

In particular, the bounds coincide for any parabolic subalgebra of $\mathfrak{g}$ simple of type A or C.

When $\mathfrak{g}$ is simple of type B or D and $\mathfrak{p}$ is a maximal parabolic subalgebra, then the bounds coincide in roughly half of the cases.

Note that this criterion is not a necessary condition for polynomiality of $Sy(\mathfrak{p})$.

For instance, when $\mathfrak{p} = \mathfrak{b}^-$, then the criterion does not hold in $\mathfrak{g}$ simple of type outside A or C, but $Sy(\mathfrak{b}^-)$ is a polynomial algebra (Joseph, 1977).

Theorem 2 (Kostant, 1963)

Assume that $\mathfrak{g}$ is a complex simple Lie algebra. Then there exists a principal $\mathfrak{sl}_2$-triple $(x,\, h,\, y)$ with $h \in \mathfrak{h}$, $x,\, y$ regular nilpotent in $\mathfrak{g}$, such that $[h,\, x] = x$ and $[h,\, y] = -y$ and we have the following.

- *Restriction of functions gives an algebra isomorphism $Y(\mathfrak{g}) \xrightarrow{\sim} R[y + \mathfrak{g}^x]$ where $R[y + \mathfrak{g}^x]$ is the algebra of regular functions on the affine variety $y + \mathfrak{g}^x$.*
- *The degree of each homogeneous generator of $Y(\mathfrak{g})$ is equal to one plus the eigenvalue of $\mathrm{ad}\, h$ on each element of a basis of $\mathfrak{g}^x$.*
- *Every G-orbit in $G(y + \mathfrak{g}^x)$ meets $y + \mathfrak{g}^x$ transversally at exactly one point and $\overline{G(y + \mathfrak{g}^x)} = \mathfrak{g}^*$ (the affine subspace $y + \mathfrak{g}^x$ is called an affine slice to the coadjoint action of $\mathfrak{g}$).*

Adapted pair

When the algebraic Lie algebra $\mathfrak{a}$ is not semisimple, there exists no principal $\mathfrak{sl}_2$-triple in general in $\mathfrak{a}$.

Definition 3 (Joseph, 2007)

An adapted pair for $\mathfrak{a}$ is a pair $(h, y) \in \mathfrak{a} \times \mathfrak{a}^*$, with h ad-semisimple, y regular in $\mathfrak{a}^*$ (that is, the coadjoint orbit $A.y$ is of minimal codimension, the index of $\mathfrak{a}$) and $(\mathrm{ad}\, h)(y) = -y$.

Remarks 4

- *Adapted pairs were constructed for all truncated (bi)parabolic subalgebras in $\mathfrak{g}$ simple of type A (Joseph, 2008).*
- *Adapted pairs need not exist (for instance for $\mathfrak{b}_\Lambda$ in $\mathfrak{g}$ simple of type B, D, E, F, G).*

An improved upper bound.

Lemma 5 (Joseph, 2008)

Assume that there exists an adapted pair $(h,\, y) \in \mathfrak{h}_\Lambda \times \mathfrak{p}_\Lambda^*$ *for the canonical truncation* $\mathfrak{p}_\Lambda$ *of* $\mathfrak{p}$ *such that* $y = \sum_{\gamma \in S} x_\gamma$ *with* $x_\gamma \in \mathfrak{g}_\gamma \setminus \{0\}$, $S \subset \Delta^+ \sqcup \Delta^-_{\pi'}$*, and that* $S_{|\mathfrak{h}_\Lambda}$ *is a basis for* $\mathfrak{h}_\Lambda^*$*.*
Then there exists an "improved upper bound" b *such that* $\mathrm{ch}\, Sy(\mathfrak{p}) \leq b$.
Moreover if equality holds, then restriction of functions gives an isomorphism of algebras $Sy(\mathfrak{p}) \xrightarrow{\sim} R[y+V]$ *where* $(\mathrm{ad}\, \mathfrak{p}_\Lambda)(y) \oplus V = \mathfrak{p}_\Lambda^*$ *and* V *is an* h*-stable vector space.*
Such an affine subspace $y + V$ *is called a Weierstrass section or an algebraic slice.*

Theorem 6 (Joseph - Shafrir, 2010)

Assume that $Sy(\mathfrak{p}) = Y(\mathfrak{p}_\Lambda)$ is a polynomial algebra in $\ell := \text{ind}\, \mathfrak{p}_\Lambda$ generators and that $\mathfrak{p}_\Lambda$ admits an adapted pair $(h, y) \in \mathfrak{h}_\Lambda \times \mathfrak{p}_\Lambda^*$.

- *Then $y + V$ is a Weierstrass section and it is also an affine slice to the coadjoint action of $\mathfrak{p}_\Lambda$ (where V is an h-stable complement of $(\mathrm{ad}\, \mathfrak{p}_\Lambda)(y)$ in $\mathfrak{p}_\Lambda^*$).*
- *If $\{m_i\}_{i=1}^{\ell}$ are the eigenvalues of $\mathrm{ad}\, h$ on a basis of V, then $\{m_i + 1\}_{i=1}^{\ell}$ are the degrees of the generators of $Y(\mathfrak{p}_\Lambda)$.*
- *In particular, $m_i \geq 0$.*

Maximal parabolic subalgebras.

This is a joint work with Polyxeni Lamprou.

From now on, we focus on *maximal* parabolic subalgebras $\mathfrak{p}$ of $\mathfrak{g}$ simple of type B or D.

That is, $\mathfrak{p} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}_{\pi'}$ with $\pi' = \pi \setminus \{\alpha_s\}$.

In this case the canonical truncation $\mathfrak{p}_\Lambda$ of $\mathfrak{p}$ is such that $\mathfrak{p}_\Lambda = \mathfrak{n}^- \oplus \mathfrak{h}' \oplus \mathfrak{n}_{\pi'}$ with $\mathfrak{h}' = \mathfrak{h} \cap [\mathfrak{p}, \mathfrak{p}]$.

By the criterion of Theorem 1, we obtain that $Sy(\mathfrak{p})$ is a polynomial algebra when s is odd (with Bourbaki's labeling).

Then it remains the case s even, which we will treat by constructing an adapted pair for $\mathfrak{p}_\Lambda$ and then use Lemma 5, showing that the improved upper bound is reached.

Definition 7

Let $\gamma \in \Delta$. An Heisenberg set Γ_γ of centre γ is a subset of Δ such that

1. $\gamma \in \Gamma_\gamma$
2. *for all $\alpha \in \Gamma_\gamma \setminus \{\gamma\}$ there exists $\alpha' \in \Gamma_\gamma$ such that $\alpha + \alpha' = \gamma$.*

Example 0.1

Let $\Delta = \sqcup \Delta_i$ be a root system, Δ_i irreducible root systems and β_i the unique highest root of Δ_i.
Take $(\Delta_i)_{\beta_i} := \{\alpha \in \Delta_i \mid (\alpha, \beta_i) = 0\}$ and decompose it into irreducible root systems Δ_{ij} with highest roots β_{ij}.
Continuing we obtain a set of strongly orthogonal positive roots β_K, indexed by elements $K \in \mathbb{N} \cup \mathbb{N}^2 \cup \cdots$.
The sets $H_{\beta_K} := \{\alpha \in \Delta_K \mid (\alpha, \beta_K) > 0\}$ are Heisenberg sets of centre β_K.

A criterion of regularity.

For $A \subset \Delta$, denote by $\mathfrak{g}_A := \bigoplus_{\alpha \in A} \mathfrak{g}_\alpha$ with $\mathfrak{g}_\alpha$ the root subspace of $\mathfrak{g}$ corresponding to α.
Let S be a subset of $\Delta^+ \sqcup \Delta^-_{\pi'}$.
We choose for all $\gamma \in S$ an Heisenberg set $\Gamma_\gamma \subset \Delta^+ \sqcup \Delta^-_{\pi'}$ such that all Γ_γ's are disjoint.
We set $O = \bigsqcup_{\gamma \in S} \Gamma^0_\gamma$ (with $\Gamma^0_\gamma = \Gamma_\gamma \setminus \{\gamma\}$) and $\mathfrak{o} = \mathfrak{g}_{-O}$.

Lemma 8

Assume that

1. $y = \sum_{\gamma \in S} x_\gamma$, *with* $x_\gamma \in \mathfrak{g}_\gamma \setminus \{0\}$ *for all* $\gamma \in S$.
2. $\Delta^+ \sqcup \Delta^-_{\pi'} = \bigsqcup_{\gamma \in S} \Gamma_\gamma \sqcup T$.
3. $|T| = \operatorname{ind} \mathfrak{p}_\Lambda$.
4. $S_{|\mathfrak{h}_\Lambda}$ *is a basis for* $\mathfrak{h}^*_\Lambda$.
5. *The restriction of* $\Phi_y : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{C}$ *defined by* $\Phi_y(x, x') = K(y, [x, x'])$ *(with* K *the Killing form of* $\mathfrak{g}$*) to* $\mathfrak{o} \times \mathfrak{o}$ *is non-degenerate.*

   Then y *is regular in* $\mathfrak{p}^*_\Lambda$ *and* (h, y) *is an adapted pair for* $\mathfrak{p}_\Lambda$ *(where* $h \in \mathfrak{h}_\Lambda$ *is defined by* $\gamma(h) = -1$ *for all* $\gamma \in S$*).*

A criterion of non-degeneracy.

We set $S = S^+ \sqcup S^- \sqcup S^m$ with S^+, resp. S^-, being the subset of S containing those $\gamma \in S$ such that $\Gamma_\gamma \subset \Delta^+$, resp. $\Gamma_\gamma \subset \Delta^-_{\pi'}$.
Set $O^\pm = \bigsqcup_{\gamma \in S^\pm} \Gamma^0_\gamma$, $O^m = \bigsqcup_{\gamma \in S^m} \Gamma^0_\gamma$ and $O = O^+ \sqcup O^- \sqcup O^m$.

Lemma 9 (Generalization of a lemma of Joseph.)

Assume that

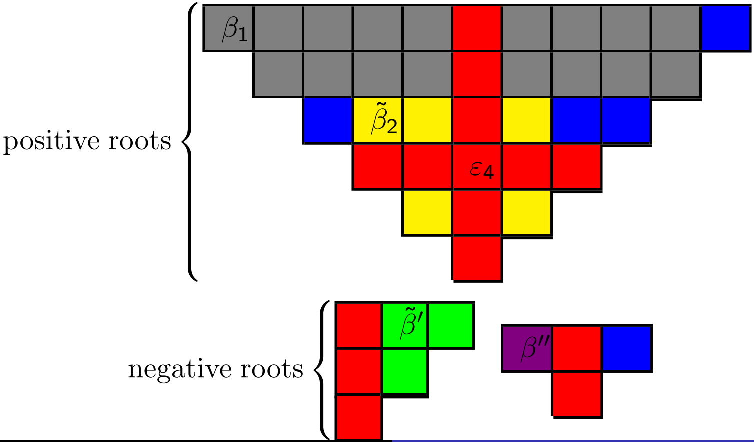
1. $y = \sum_{\gamma \in S} x_\gamma$, *with* $x_\gamma \in \mathfrak{g}_\gamma \setminus \{0\}$ *for all* $\gamma \in S$.
2. $S_{|\mathfrak{h}_\Lambda}$ *is a basis for* $\mathfrak{h}^*_\Lambda$.
3. *If* $\alpha \in \Gamma^0_\gamma$, *with* $\gamma \in S^+$, *is such that there exists* $\beta \in O^+$, *with* $\alpha + \beta \in S$, *then* $\beta \in \Gamma^0_\gamma$ *and* $\alpha + \beta = \gamma$.
4. *If* $\alpha \in \Gamma^0_\gamma$, *with* $\gamma \in S^-$, *is such that there exists* $\beta \in O^-$, *with* $\alpha + \beta \in S$, *then* $\beta \in \Gamma^0_\gamma$ *and* $\alpha + \beta = \gamma$.
5. *Additional technical conditions for all* $\alpha \in O$ *such that there exists* $\beta \in O^m$, *with* $\alpha + \beta \in S$.

Then the restriction of Φ_y *to* $\mathfrak{o} \times \mathfrak{o}$ *is non-degenerate.*

Example for B_6 with $\pi' = \pi \setminus \{\alpha_4\}$.

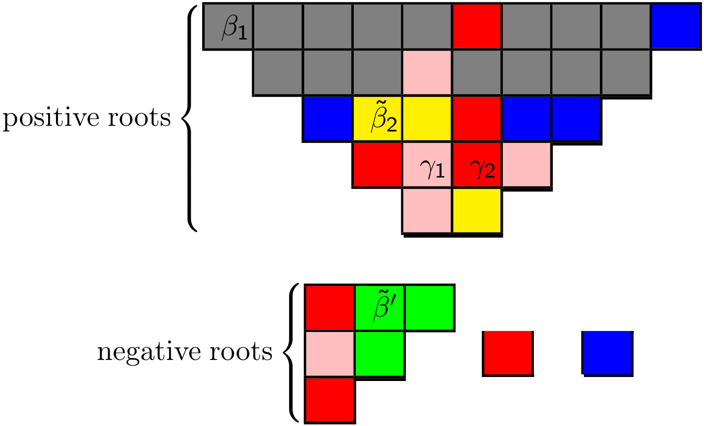
We choose

$$S = \{\beta_1 = \varepsilon_1 + \varepsilon_2,\ \tilde{\beta}_2 = \beta_2 - \alpha_4 = \varepsilon_3 + \varepsilon_5,\ \varepsilon_4,$$
$$-\tilde{\beta}' = -\beta' + \alpha_3 = \varepsilon_3 - \varepsilon_1, -\beta'' = -\varepsilon_5 - \varepsilon_6\}$$
$$T = \{\alpha_1,\ \beta_2,\ \alpha_3,\ \alpha_3 + \alpha_4,\ -\alpha_5\} \text{ (roots in blue)}$$



We choose

$$S = \{\beta_1 = \varepsilon_1 + \varepsilon_2,\ \tilde{\beta}_2 = \beta_2 - \alpha_4 = \varepsilon_3 + \varepsilon_5,$$
$$-\tilde{\beta}' = -\beta' + \alpha_3 = \varepsilon_3 - \varepsilon_1,\ \gamma_1 = \varepsilon_4 + \varepsilon_6,\ \gamma_2 = \varepsilon_4 - \varepsilon_6\}$$
$$T = \{\alpha_1,\ \beta_2,\ \alpha_3,\ \alpha_3 + \alpha_4,\ -\alpha_5\} \text{ (roots in blue)}$$



Polynomiality of $Sy(\mathfrak{p})$ for a maximal parabolic subalgebra $\mathfrak{p}$.

Theorem 10 (F-M - Lamprou, arXiv 2017)

Let $\mathfrak{p}$ be any maximal parabolic subalgebra of a simple Lie algebra $\mathfrak{g}$ of type B, D or E_6. Then $Sy(\mathfrak{p})$ is a polynomial algebra and there exists a Weierstrass section (given by an adapted pair for the canonical truncation $\mathfrak{p}_\Lambda$) which is also an affine slice to the coadjoint action of $\mathfrak{p}_\Lambda$.

Remarks 11

- *For the cases when lower and upper bounds* $\mathrm{ch}\,\mathscr{A}$ *and* $\mathrm{ch}\,\mathscr{B}$ *of Theorem 1 coincide, we have constructed an adapted pair for* $\mathfrak{p}_\Lambda$ *(F-M - Lamprou, 2016) (in this case, the subset* $S^m = \emptyset$*). Then we apply Theorem 6.*
- *When lower and upper bounds of Theorem 1 donnot coincide, we have also constructed an adapted pair for* $\mathfrak{p}_\Lambda$ *(in this case, adapted pairs are more complicated to construct than in the previous case, and in particular the subset* $S^m \neq \emptyset$ *and then the additional conditions of Lemma 9 are needed).*
  Then we have computed the improved upper bound b *and have shown that* $b = \mathrm{ch}\,\mathscr{A}$.
  Applying Lemma 5, we conclude that $Sy(\mathfrak{p})$ *is a polynomial algebra, whose degrees of generators can be computed by Theorem 6.*
- *For type* E_6*, when lower and upper bound donnot coincide, an adapted pair was found by computer calculations.*

Further.

For other parabolic subalgebras when the bounds in Theorem 1 donnot coincide, we hope to find an adapted pair $(h,\ y)$ for the canonical truncation $\mathfrak{p}_{\Lambda}$.

The strategy.

Theorem 12 (Joseph - F-M, 2016)

Every second element of an adapted pair for the canonical truncation $\mathfrak{p}_\Lambda$ of a (bi)parabolic subalgebra of $\mathfrak{g}$ simple of type A is the restriction to $\mathfrak{p}_\Lambda$ of a regular nilpotent element of $\mathfrak{g}$.

In particular, this gives an element w of the Weyl group of $\mathfrak{g}$ which sends the set π of simple roots of $\mathfrak{g}$ to a new system of simple roots.

The previous theorem is motivated by the following theorem.

Theorem 13 (Joseph, 2011)

Thanks to an element of the Weyl group which is roughly the "square root" of the longest element of the Weyl group of $\mathfrak{g}$, there is a combinatorial receipt which gives Weierstrass sections for the Borel subalgebra of $\mathfrak{g}$ (for almost all simple Lie algebras $\mathfrak{g}$).

Example

Now assume that $\mathfrak{g}$ is simple of type B_n and embed $\mathfrak{g}$ in $\mathfrak{sl}_{2n+1}$.
Let $\mathfrak{p}$ be a maximal parabolic subalgebra of $\mathfrak{g}$ and consider the adapted pair $(h,\ y)$ for $\mathfrak{p}_\Lambda$ we have constructed. Then applying the method of Theorem 12 in type A, one may compute a regular nilpotent element y^* of $\mathfrak{g}$ such that y is its restriction to $\mathfrak{p}_\Lambda$.
This provides a new set of simple roots π^* and then an element of the Weyl group of $\mathfrak{g}$ which sends π to π^*.
One may hope to generalize such an element of the Weyl group to other parabolic subalgebras of $\mathfrak{g}$, which are not maximal and then get an adapted pair for $\mathfrak{p}_\Lambda$, for example when $\pi' = \pi \setminus \{\alpha_s,\ \alpha_{s+2}\}$, with s even.