Algebraic and geometrical description of the fibers of the Mumford system

Yasmine Fittouhi

Yasmine.Fittouhi@math.univ-poitiers.fr

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We will use the moto of Julius Caesar:

"Divide and rule" ("divide ut regnes").

Overview

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To define Mumford system of order g, we need to define three object:

- Phase space,
- Family of vectors fields (Poisson structure),
- Momentum map.

Phase space

We fix an integer $g > 1$. We work on the field $\mathbb{C}$

Mumford system of order g progress on **complex affine space** named M_g with $u_0, \ldots, u_{g-1},\ v_0, \ldots, v_{g-1},\ w_{0,} \ldots, w_g$ its affine coordinated.

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The phase space M_g is defined as follow[1]:

$$M_g := \left\{ \begin{pmatrix} v(x) & u(x) \\ w(x) & -v(x) \end{pmatrix} \in M_{2,2}(\mathbb{C}[x])$$

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$$u(x) \quad = \quad x^g + u_{g-1}x^{g-1} + u_{g-2}x^{g-2} + \cdots + u_0,$$

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$$\begin{aligned} u(x) &= x^g + u_{g-1}x^{g-1} + u_{g-2}x^{g-2} + \cdots + u_0, \\ v(x) &= v_{g-1}x^{g-1} + v_{g-2}x^{g-2} + \cdots + v_0, \end{aligned}$$

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$$\{u(x), u(y)\} = \{x^g + u_{g-1}x^{g-1} + \cdots + u_0, y^g + u_{g-1}y^{g-1} + \cdots + u_0\},$$

$$\{u(x), u(y)\} = \sum_{i,j=0}^{g-1} \{u_i, u_j\} x^i y^j.$$

Poisson structure

Definition

The Poisson structures on the space M_g are codified by these equations

$$\begin{aligned} \{u(x),u(y)\} &= \{v(x),v(y)\} = 0, \\ \{u(x),v(y)\} &= \frac{u(x)-u(y)}{x-y}, \\ \{u(x),w(y)\} &= -2\frac{v(x)-v(y)}{x-y}, \\ \{v(x),w(y)\} &= \frac{w(x)-w(y)}{x-y} - u(x), \\ \{w(x),w(y)\} &= 2(v(x)-v(y)). \end{aligned} \tag{1}$$

The equations (0.1) enable us to know the Poisson bracket for the coordinates functions $u_{g-1},\dots,u_0$, $v_{g-1},\dots,v_0$, $w_g,\dots,w_0$.

Momentum map

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Let's note by H_g the affine space of dimension $2g+1$ define by

$$H_g = \{x^{2g+1} + a_{2g}x^{2g} + \cdots + a_0 \mid (a_{2g}, \ldots, a_0) \in \mathbb{C}^{2g+1}\}.$$

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The momentum map is noted **H** is define from M_g to H_g:

$$\begin{array}{llcl} \mathbf{H}: & M_g & \longrightarrow & H_g \\ & A(x) = \begin{pmatrix} v(x) & u(x) \\ w(x) & -v(x) \end{pmatrix} & \longrightarrow & -\det(A(x)) = v(x)^2 + u(x)w(x). \end{array}$$

- The composantes of **H** defines $2g+1$ polynomials functions of M_g denoted by $h_0, \ldots, h_{2g+1}$ defined by,

$$\mathbf{H}(A(x)) = \sum_{i=0}^{2g+1} h_i(A(x))x^i. \tag{2}$$

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- For $y \in \mathbb{C}$, denoted by $(\mathbf{H}_y)_{y\in\mathbb{C}}$ polynomials functions defined by

$$\begin{array}{cccc} \mathbf{H}_y : & M_g & \longrightarrow & \mathbb{C} \\ & A(x) & \longmapsto & \mathbf{H}(A(x))|_{x=y} = -\det(A(y)). \end{array}$$

Family of vectors fields

The two families of polynomials functions $(h_i)_{i=0,\ldots,2g}$ and $(\mathbf{H}_y)_{y\in\mathbb{C}}$ enable us to define a family of hamiltonian vectors fields.

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Proposition

Let $A(x) = \begin{pmatrix} v(x) & u(x) \\ w(x) & -v(x) \end{pmatrix} \in M_g$*. For each* $y \in \mathbb{C}$ *and* $0 \leqslant i \leqslant 2g$ *the vecteurs fields* D_y *at the point* $A(x)$ *is define by the equation :*

$$D_y|_{A(x)} = \left[A(x), -\frac{A(y)}{x-y} - \begin{pmatrix} 0 & 0 \\ u(y) & 0 \end{pmatrix} \right],$$

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$$D_i|_{A(x)} = \left[A(x), \left[\frac{A(x)}{x^{i+1}}\right]_+ - \begin{pmatrix} 0 & 0 \\ u_i & 0 \end{pmatrix}\right], \tag{3}$$

where $\left[\frac{A(x)}{x^{i+1}}\right]_+$ *is the polynomial part.*

Remarks

- We note, from the equation (3), we have $\left[\frac{A(x)}{x^{i+1}}\right]_+ - \begin{pmatrix} 0 & 0 \\ u_i & 0 \end{pmatrix} = 0$ for all $i \geqslant g$, therefore the functions $h_g, \ldots, h_{2g}$ are Casimir functions for the Poisson structure $\{\cdot, \cdot\}$.

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By definition, $\mathbf{H}_y(A(x)) = \mathbf{H}(A(x))|_{x=y}$, therefore $\mathbf{H}_y = \sum_{i=0}^{2g+1} y^i h_i$, we get

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- The space

$$\mathcal{U} = \{A \in M_g \text{such that } D_0|_A \wedge \cdots \wedge D_{g-1}|_A \neq 0\}$$

is not empty, and it is a Zariski an open set dense in M_g.

Proposition

The system $(M_g, \{\cdot,\cdot\}, \mathbf{H})$ *is integrable in sens of Liouville of rank* g*. This system is named Mumford system of order* g*.*

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When

$$\Delta(h) \neq 0 \implies M_g(h) \sim \mathrm{Jac}(C) - \Theta,$$

where C is curve with genus g.
Since $Jac(C)$ is isomorphic to a torus, it was natural to find the solution as theta function since it has a periodicity.

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$$\implies 0 = D_0|_A \implies \dim < D_{g-1}|_A, \cdots, D_0|_A >< g$$

conclude

- If $A(x) = \begin{pmatrix} v(x) & u(x) \\ w(x) & -v(x) \end{pmatrix} \in M_g(h)$ and

$$\mathrm{GCD}(u, v, w) = Q$$

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Stratification

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Definition

Let $(I, \leqslant)$ a partially ordered set. *Stratification* of affine variety V, is a partition of V by a family $(S_i)_{i\in I}$ of quasi-affines variety such that :
For each $i \in I$, the Zariski closure $\overline{S_i}$ of S_i is

$$\overline{S_i} = \bigsqcup_{j\leqslant i} S_j, \text{ (disjoint union).} \tag{4}$$

The sets S_i are called *stratum*.

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$$M_{g,Q}(h) = \left\{ A(x) = \begin{pmatrix} v(x) & u(x) \\ w(x) & -v(x) \end{pmatrix} \in M_g(h) \text{ such that } \right.$$

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Remark, If

$$A(x) \in M_{g,Q}(h) \implies \dim < D_{g-1}|_A, \cdots, D_0|_A >= g - \deg(Q) = g - j$$

Theorem

The family $(M_{g,Q}(h))_{Q\in\mathbb{C}[x]}$ *is stratification.*

Relation between Any stratum and Mumford system of lower degree

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Any stratum of the stratification $(M_{g,Q}(h))_{Q\in\mathbb{C}_h[x]}$ is isomorphic to a stratum of an Mumford system of lower degree of the form $(M_{g-\deg(Q),1}(\frac{h}{Q^2}))_{Q\in\mathbb{C}_h[x]}$.

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Let $h \in H_g$.
Let us denote h monic polynomial such that $h(x) = P^2(x)h'(x)$, with the discriminant of h' is non zero and $\deg h' = 2g' + 1$.

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The factorization of the polynomial of P is the following:

$$P(x) = \prod_{i=1}^{k} (x - a_i)^{\ell_i}. \tag{5}$$

The roots $a_i \in \mathbb{C}$ of $P(x)$ are all distinct,

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Mumford system is an Algebraic complete integrability (a.c.i).

Solutions

Since

$$\text{Jac}(C') \times \mathbb{C}^{*n} \times \mathbb{C}^m.$$

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The solutions should look like functions of theta function plus something

else no-periodic.