Complex reflection groups and Diagrammatic Cherednik algebras

July 20, 2017

Section 1

Decomposition numbers of reflection groups

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- It is a highest weight category and is obtained from GL_h via Ringel duality.
- Our main interest is in the graded decomposition numbers,

$$d_{\lambda\mu}(t) = \sum_{k\in\mathbb{Z}} [S(\lambda) : D(\mu)\langle k\rangle] t^k$$

for $\lambda, \mu \in \mathcal{P}_n(h)$.

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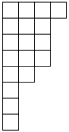

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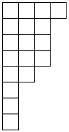

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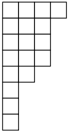
$\mathcal{P}_{19}(4) \to \mathbb{E}_4$  $\mapsto 9\varepsilon_1 + 5\varepsilon_2 + 4\varepsilon_3 + \varepsilon_4$

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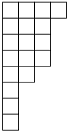

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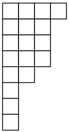

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Question

Does this generalise to other complex reflection groups?

Section 2

The super-strong linkage principle for symmetric groups

Sneak peak.......

The super-strong linkage principle (B.–Cox)

For every $k \in \mathbb{Z}$ we have that

$$[S(\lambda) : D(\mu)\langle k\rangle] \leq |\{\mathsf{s} \mid \mathsf{s} \in \mathrm{Path}^+(\lambda, \mathsf{t}^\mu), \deg(\mathsf{s}) = k\}|$$

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- New upper bounds for graded decomposition numbers of symmetric groups in terms of dominant paths in our alcove geometry.
- This generalises to the cyclotomic Hecke algebras (i.e. those of type $G(\ell, 1, n)$ — the deformations of $(\mathbb{Z}/\ell\mathbb{Z}) \wr \mathfrak{S}_n$).

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The strong linkage principle (Humphreys (see also Jantzen, Andersen))

Let $\lambda, \mu \in \mathcal{P}_n(h)$ *and* $p > h$. *If* $d_{\lambda\mu}(t) \neq 0$ *then* $\lambda \uparrow \mu$.

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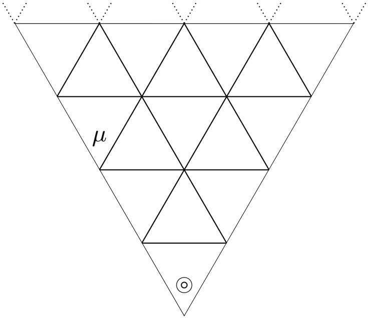

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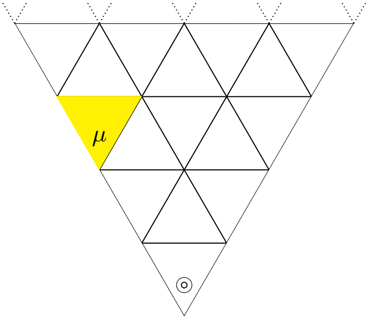

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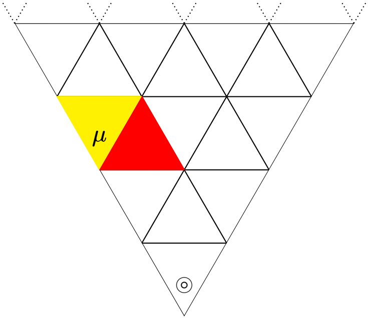

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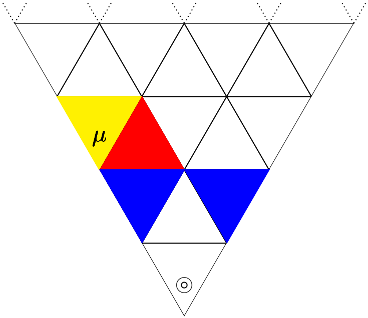

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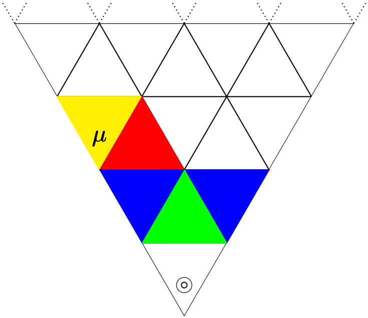

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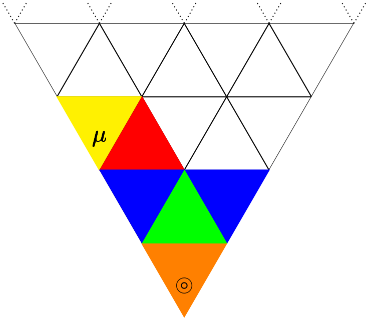

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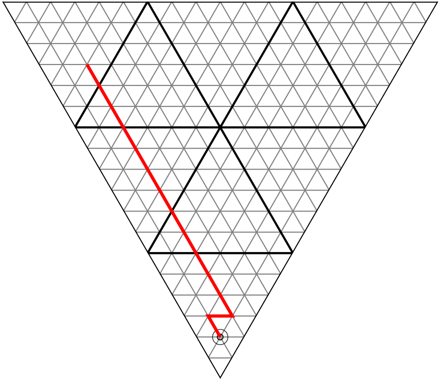

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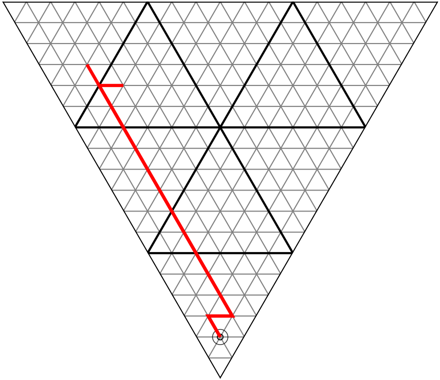

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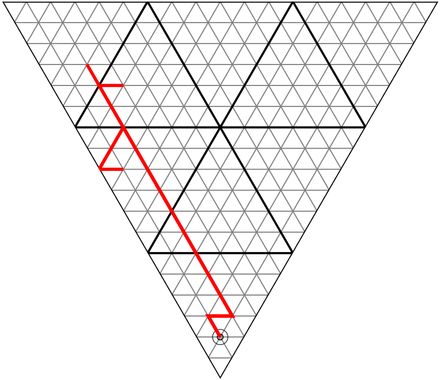

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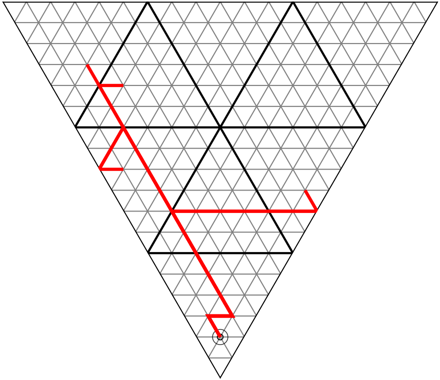

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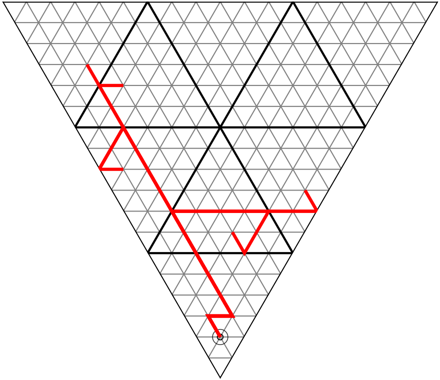

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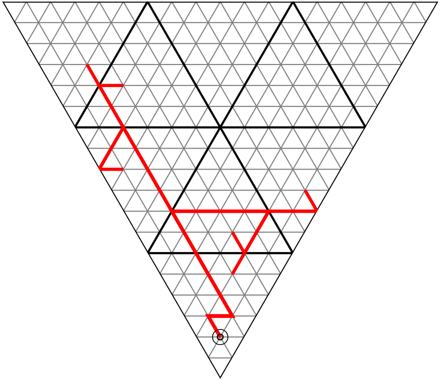

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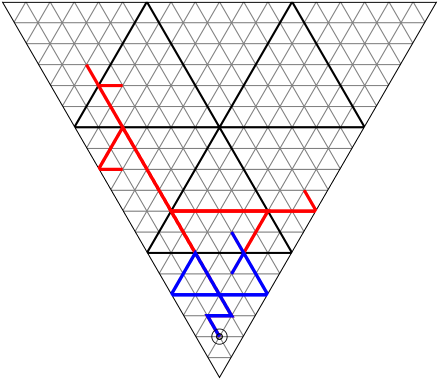

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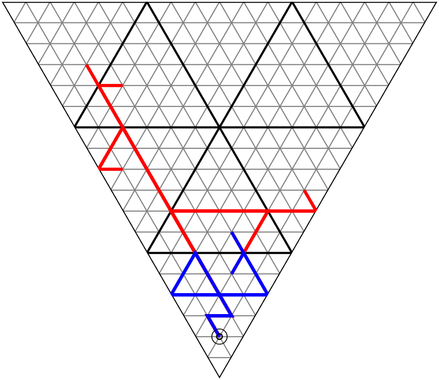

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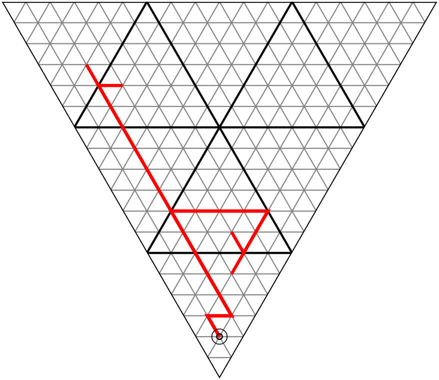

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Theorem (B.–Cox)

There is a quotient of the "diagrammatic Cherednik algebra" which possesses a graded cellular basis

$$\{C_{\mathsf{st}} \mid \mathsf{s} \in \mathrm{Path}_n^+(\lambda, \mathsf{t}^\mu), \mathsf{t} \in \mathrm{Path}_n^+(\lambda, \mathsf{t}^\nu), \lambda, \mu, \nu \in \mathcal{P}_n(h)\}.$$

We show that this quotient of the "diagrammatic Cherednik algebra" is Morita equivalent to the relevant quotient of the symmetric group and hence...

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Corollary (B.–Cox)

Let $p > h$. For every $k \in \mathbb{Z}$ we have that

$$[S(\lambda) : D(\mu)\langle k\rangle] \leq |\{\mathsf{s} \mid \mathsf{s} \in \mathrm{Path}^+(\lambda, \mathsf{t}^\mu), \deg(\mathsf{s}) = k\}|$$

for $\lambda, \mu \in \mathcal{P}_n(h)$.

- Let $\Bbbk$ be an arbitrary field and $e = 6$. For $\mu = (2, 1^{12})$, there are only 4 dominant paths. These are the paths which terminate at the points

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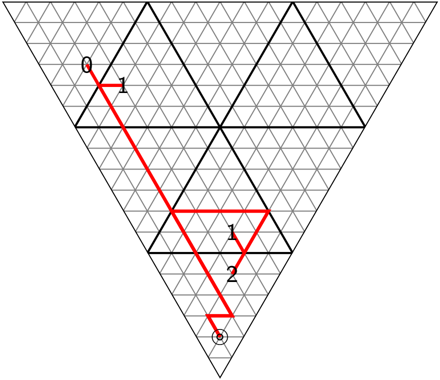

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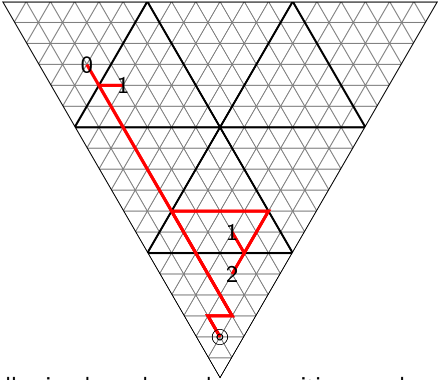

- We have the following bounds on decomposition numbers,

$$d_{(2,1^{12}),(2,1^{12})}(t) \leq 1 \quad d_{(2^2,1^{10}),(2,1^{12})}(t) \leq t^1$$
$$d_{(3^3,2^2,1),(2,1^{12})}(t) \leq t^2 \quad d_{(3^2,2^3,1^2),(2,1^{12})}(t) \leq t^1.$$

and $d_{\lambda\mu}(t) = 0$ otherwise. These bounds are sharp!

Section 3

Decomposition numbers for cyclotomic quiver Hecke algebras

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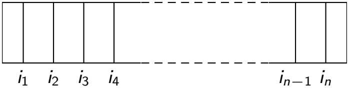
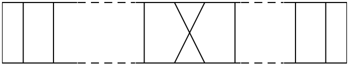
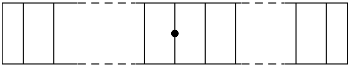
The quiver Hecke algebra, $H_n^{\Bbbk}(\underline{\sigma})$, is generated by the idempotents

$e(\underline{i}) =$ 

for $i \in \mathbb{Z}/e\mathbb{Z}$, along with the elements

$\psi_r =$ 

$y_r =$ 

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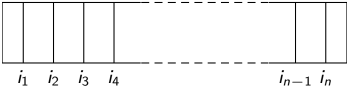
$e(\underline{i}) =$ 

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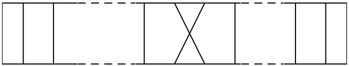
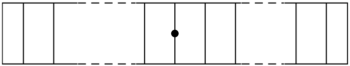
$\psi_r =$ 

$y_r =$ 

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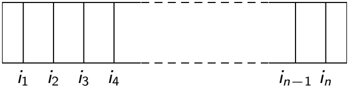

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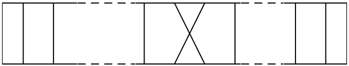

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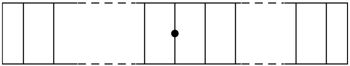

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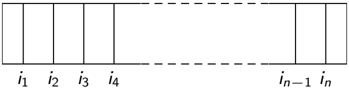
$e(\underline{i}) =$ 

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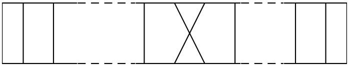
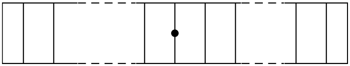
$\psi_r =$ 

$y_r =$ 

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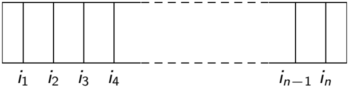
$e(\underline{i}) =$ 

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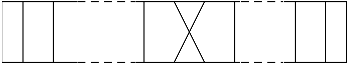
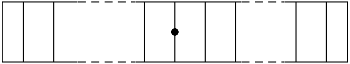
$\psi_r =$ 

$y_r =$ 

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Section 4

Multipartition combinatorics and cellularity

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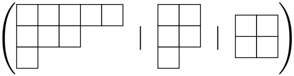

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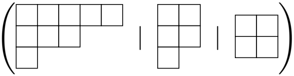
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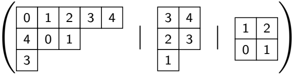

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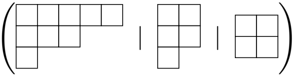

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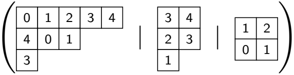

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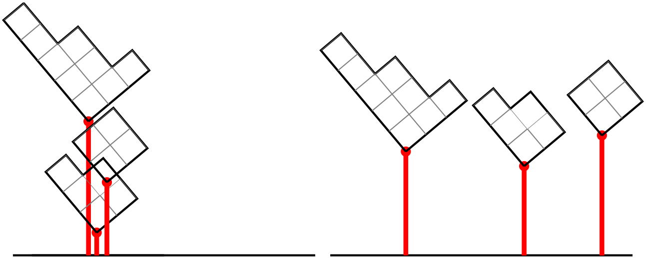

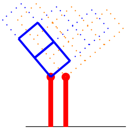

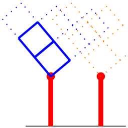

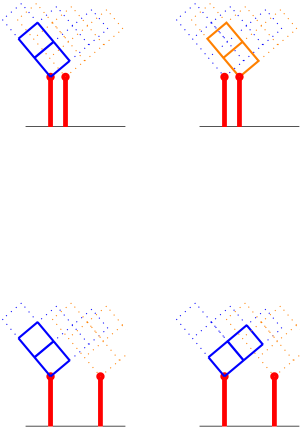

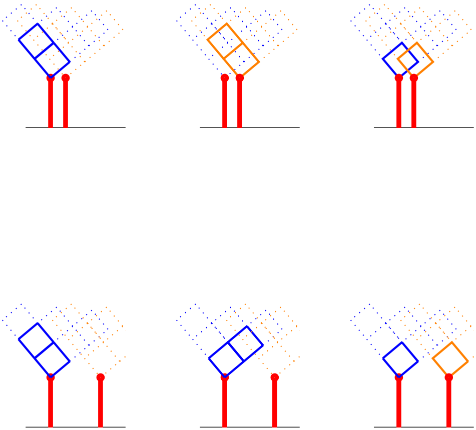

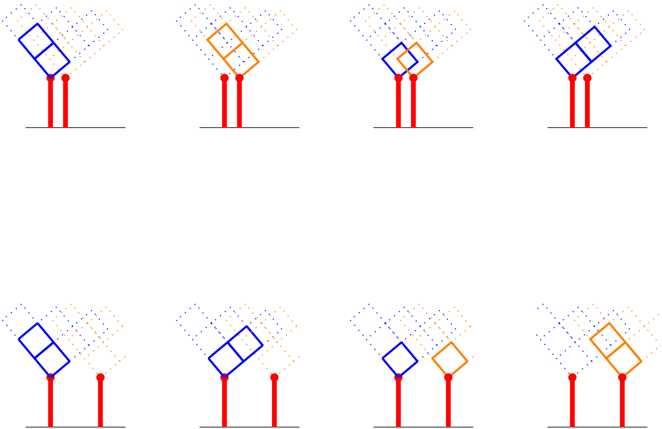

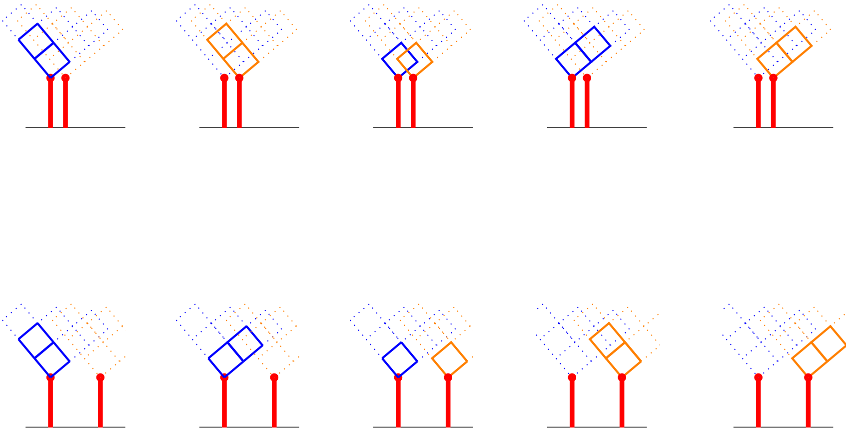

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Theorem (B., and implicit over a field by Webster)

The algebra $H_n^{\mathbb{Z}}(\underline{s})$ of type $G(\ell,1,n)$ free as a $\mathbb{Z}$-algebra with graded cellular basis

$$\{c^{\sigma}_{\mathsf{st}} \mid \lambda \in \mathcal{P}^\ell_n, \mathsf{s},\mathsf{t} \in \mathrm{Std}(\lambda)\}$$

with respect to the $\underline{\sigma}$-dominance order on $\mathcal{P}^\ell_n$ and the $\underline{\sigma}$-grading on standard tableaux.

Section 5

The graded decomposition matrices of Hecke and diagrammatic Cherednik algebras

- For each weighting, $\underline{\sigma} \in \mathbb{Z}^\ell$, the set of simples

$$\{D^{\underline{\sigma}}(\lambda) \mid \lambda \in \Sigma \subseteq \mathcal{P}^\ell_n\}$$

can be constructed from the set of cell modules,

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- The (graded) decomposition matrices are uni-triangular over $\Bbbk$ a field.

Section 6

Modular decomposition numbers of Ariki–Koike and diagrammatic Cherednik algebras

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- What about higher levels?
- For arbitrary $\underline{\sigma}$ this is impossible (it includes the level 1 problem)

Question

Can we understand decomposition matrices of Hecke and diagrammatic Cherednik algebras over arbitrary fields via Kazhdan–Lusztig theory?

- Over $\mathbb{C}$, yes! (Rouquier–Shan–Varagnolo–Vasserot, Losev, Webster.)
- For $\ell = 1$ and arbitrary $\Bbbk$, no.
- To fix this, we restrict to
  - the subcategory labelled by partitions with at most h columns
  - fields of characteristic $p \gg h$
- What about higher levels?
- For arbitrary $\underline{\sigma}$ this is impossible (it includes the level 1 problem)
- But what about for a nice choice of weighting....

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- Therefore the (square) decomposition matrix of $\mathbb{Q}_{\ell,h,n}(\underline{s})$ appears as a submatrix of that of $H_n^{\Bbbk}(\underline{s})$.
- For example $\mathbb{Q}_{1,h,n}(\underline{s})$ is the image of $\Bbbk\mathfrak{S}_n$ in $\mathrm{End}((\Bbbk^n)^{\otimes r})$.

- We want to show that the representation theory of $\mathbb{Q}_{\ell,h,n}(\underline{s})$ is controlled by an alcove geometry of type

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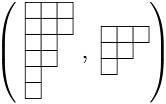
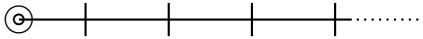
- For example

$\mathcal{P}^2_{18}(3) \to \mathbb{E}_6$  $\mapsto 6\varepsilon_1 + 4\varepsilon_2 + 2\varepsilon_3 + 3\varepsilon_5 + 2\varepsilon_6 + \varepsilon_7$

Examples of the geometries we see

$h = 2$
$\ell = 1$



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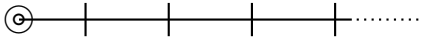

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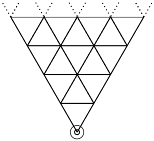

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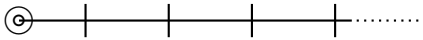

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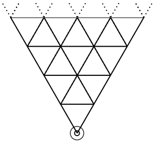

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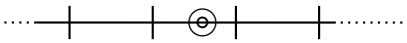

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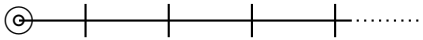

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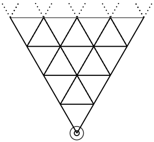

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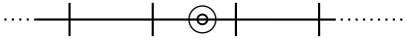

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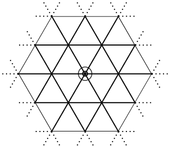

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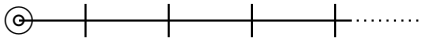

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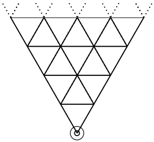

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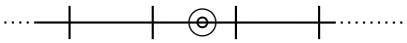

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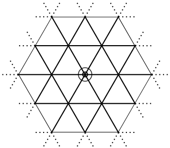

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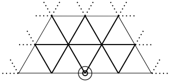

Section 7

Generic Behaviour and super-strong linkage

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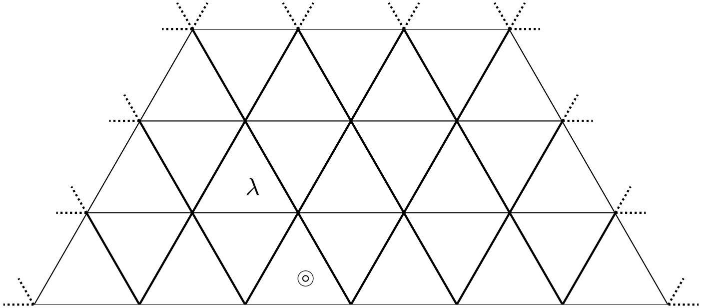
The set of coloured triangles is "generic".



Theorem (Generic Behaviour (B.–Cox))

For λ and μ generic points we have that

$$\dim_{\Bbbk}(\mathrm{Hom}_{\mathbb{Q}_{\ell,h,n}(\underline{s})}(S(\mu), S(\lambda))) = t^{\ell(\mu)-\ell(\lambda)} + \dots$$

where the other terms are of strictly smaller degree.

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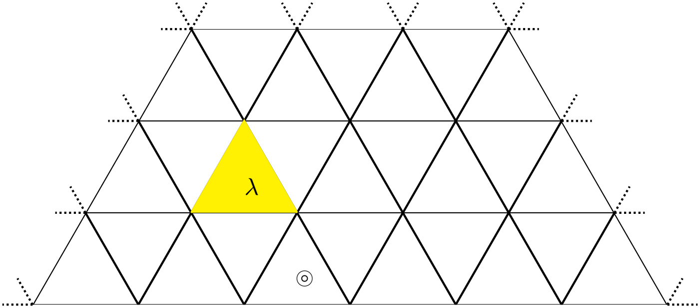

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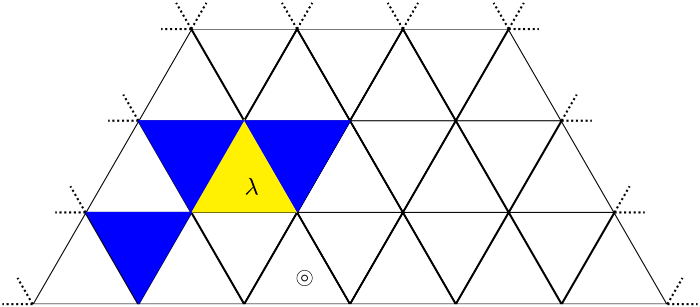

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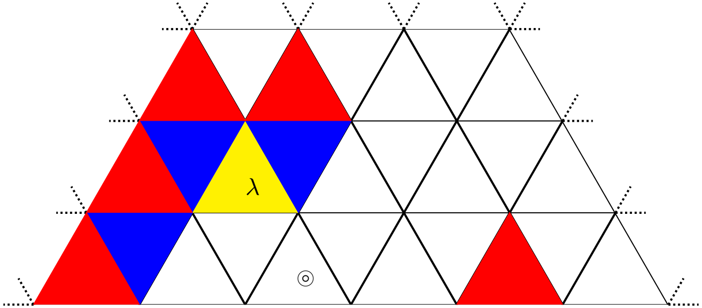

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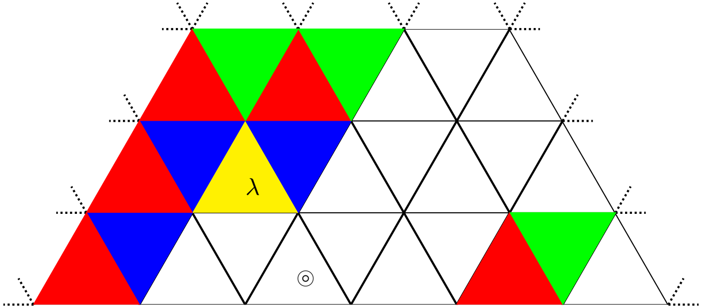

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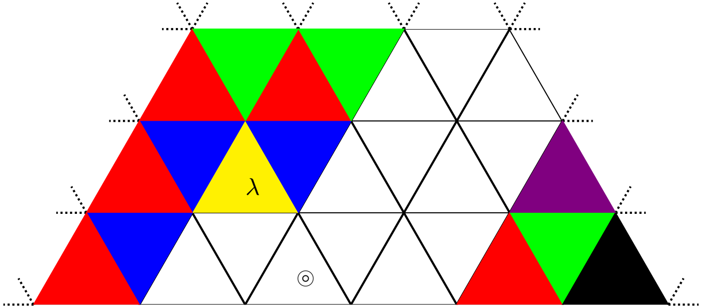

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Section 8

A conjecture for modular decomposition numbers

Theorem (B.–Cox, B.–Cox–Speyer)

Over $\mathbb{C}$, the graded decomposition numbers of $\mathbb{Q}_{\ell,h,n}(\underline{s})$ are

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Corollary (Martin–Woodcock conjecture, B.)

The decomposition matrix of the $H_n^{\mathbb{C}}(\underline{s})$ has a square submatrix with entries given by the **non-parabolic** *Kazhdan Lusztig polynomials of type $\widehat{A}_{h-1}$.*

Conjecture (Generalized Lusztig-conjecture)

Let $\mathbb{F}$ be a field of characteristic $p \gg h\ell$. The decomposition numbers are

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- $\Bbbk = \mathbb{C}$ is the complex field;
- $\ell = 1$ and $\mathrm{char}(\Bbbk) = p \gg h$ (Riche–Williamson).

Section 9

An example

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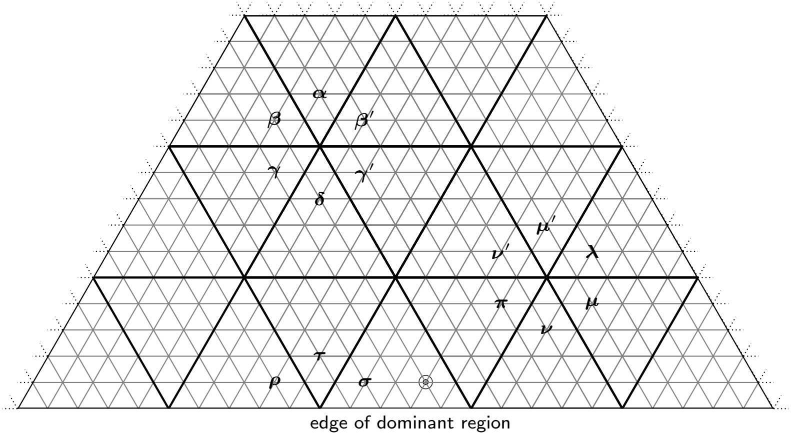

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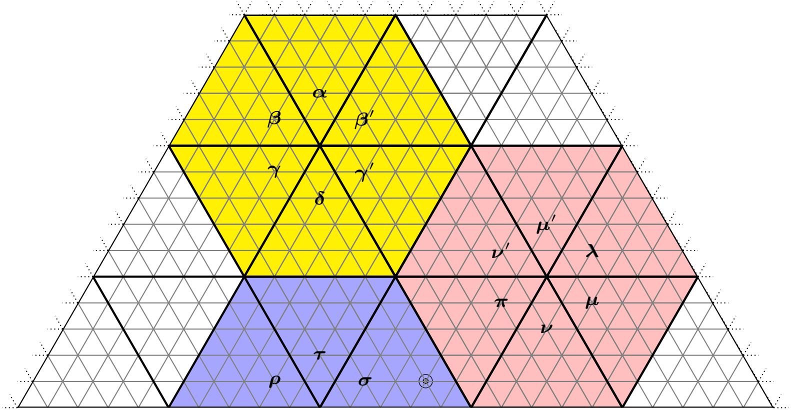


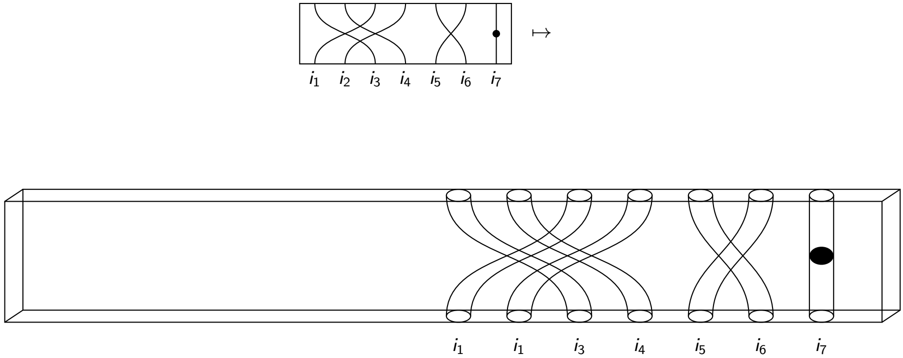
edge of dominant region

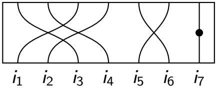
	α	β	β'	γ	γ'	δ	λ	μ	μ'	ν	ν'	π	ρ	τ	σ
α															
β															
β'				M											
γ															
γ'															
δ															
λ															
μ															
μ'				t^1M						M					
ν															
ν'															
π															
ρ	0	t^2	t^2	t	0	0	0	0	t	0	0	0	1		
τ	t^2	t^3	t	t^2	0	t	0	0	t^2	0	t	0	t	1	
σ	t^3	0	t^2	t^3	t	t^2	0	0	0	0	t^2	t	0	t	1

where M records the (non-parabolic) Kazhdan–Lusztig polynomials of type A_2.

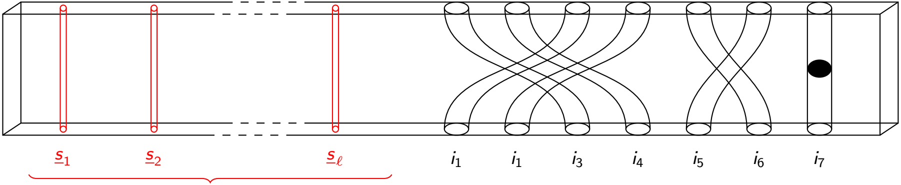
Section 10

The diagrammatic Cherednik algebras


i_1 i_2 i_3 i_4 i_5 i_6 i_7
$\mapsto$
i_1 i_1 i_3 i_4 i_5 i_6 i_7



$\mapsto$



$\underline{\sigma} \in \mathbb{Z}^\ell$ the "weighting" gives the x-coordinates

Given a weighting $\underline{\sigma} \in \mathbb{Z}^{\ell}$, we have a diagrammatic Cherednik algebra $A_n(\underline{\sigma})$ spanned by all possible tubular diagrams:

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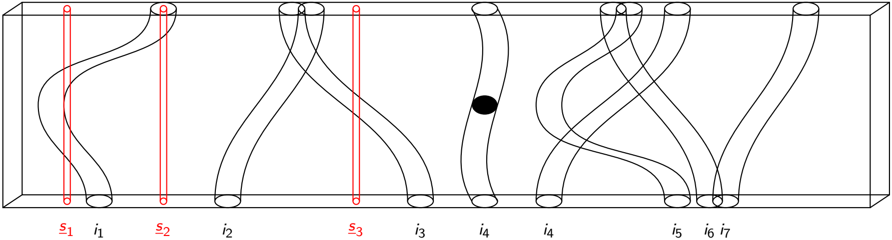

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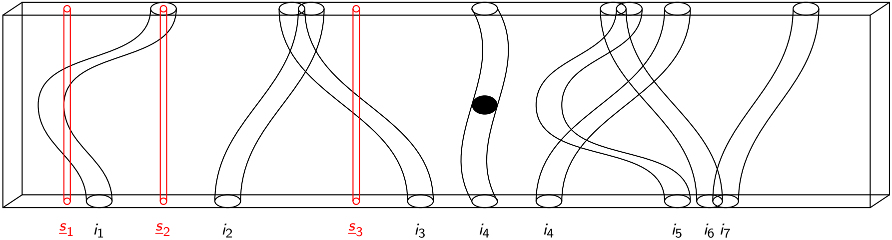

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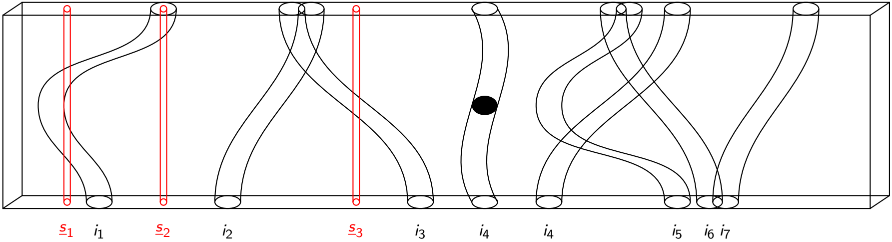

subject to the same relations.

For every $\underline{\sigma} \in \mathbb{Z}^{\ell}$ such that $\underline{\sigma} \mapsto \sigma$, the Hecke algebra, $H_n(\underline{s})$, is a subalgebra of $A_n(\underline{\sigma})$.