On the mixed scalar curvature of almost multi-product manifolds

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Introduction

We consider a smooth connected n-dimensional manifold M endowed with a pseudo-Riemannian metric $g = \langle\cdot,\,\cdot\rangle$ and $k > 2$ pairwise orthogonal n_i-dimensional non-degenerate distributions $\mathcal{D}_i$ (subbundles of TM) with $\sum n_i = n$, see [1, 2, 3].
Such $(M, g; \mathcal{D}_1, \ldots, \mathcal{D}_k)$ will be called a **Riemannian almost multi-product manifold**.

It appears in studies of the curvature and Einstein equations on multiply twisted and multiply warped products, in the theory of webs composed of different foliations, and Dupin hypersurfaces (which have a constant number of different principal curvatures).

We define the mixed scalar curvature of $(M, g; \mathcal{D}_1, \ldots, \mathcal{D}_k)$ and generalize results of the following two problems concerning this kind of curvature of almost product manifolds (i.e., $k = 2$):

- integral formulas and splitting theorems.
- variation formulas and the Einstein-Hilbert type action.

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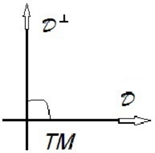
Example 1 (An averaged mixed sectional curvature)

The **mixed scalar curvature of a pair of distributions** $(\mathcal{D}, \mathcal{D}^\perp)$ on a pseudo-Riemannian manifold (M, g) is given by

$$\mathrm{S}_{\mathcal{D},\mathcal{D}^\perp} = \sum_{E_a \in \mathcal{D},\, E_b \in \mathcal{D}^\perp} \varepsilon_a\, \varepsilon_b\, K(E_a, E_b),$$



$K(E_a, E_b) = \langle R_{E_a,E_b} E_a, E_b\rangle$ – the **mixed sectional curvature**, $\{E_1, \ldots, E_n\}$ – a local **adapted** orthonormal frame on M, $E_a \in \mathcal{D}$ $(1 \le a \le \dim\mathcal{D})$, $\quad E_b \in \mathcal{D}^\perp$ $(\dim\mathcal{D} < b \le \dim M)$ and $\varepsilon_c = \langle E_c, E_c\rangle \in \{-1, 1\}$.

If one of distributions is spanned by a unit vector field N, i.e., $\langle N, N\rangle = \varepsilon_N \in \{-1, 1\}$, then

$$\mathrm{S}_{\mathcal{D},\mathcal{D}^\perp} = \varepsilon_N\, \mathrm{Ric}_{N,N},$$

$\mathrm{Ric}_{N,N} = \sum_{E_b \perp N} \varepsilon_b\, K(N, E_b)$ – **Ricci curvature** in N-direction.

We use a local **adapted** orthonormal frame $\{E_1, \ldots, E_n\}$ on M such that $\{E_1, \ldots, E_{n_1}\} \subset \mathcal{D}_1$, $\{E_{n_{i-1}+1}, \ldots, E_{n_i}\} \subset \mathcal{D}_i$ for $2 \le i \le k$. All quantities defined below using such frame do not depend on the choice of this frame.

Again, the **mixed scalar curvature of** $(M, g; \mathcal{D}_1, \ldots, \mathcal{D}_k)$ is defined as an averaged mixed sectional curvature.

Definition 2 ([2])

The **mixed scalar curvature** of $(M, g; \mathcal{D}_1, \ldots, \mathcal{D}_k)$ is the following function on M (averaging for each pair $(\mathcal{D}_i, \mathcal{D}_j)$):

$$\mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} = \sum\nolimits_{i<j} \mathrm{S}(\mathcal{D}_i, \mathcal{D}_j),$$

where $\mathrm{S}(\mathcal{D}_i, \mathcal{D}_j) = \sum\limits_{n_{i-1}<a\le n_i,\ n_{j-1}<b\le n_j} \varepsilon_a\, \varepsilon_b\, K(E_a, E_b).$

Example: multiply twisted (and warped) products

Definition 3

A **multiply twisted product** $F_0 \times_{u_1} F_1 \times \ldots \times_{u_k} F_k$ of Riemannian manifolds $(F_0, g_{F_0}), \ldots, (F_k, g_{F_k})$ is the product $M = F_0 \times \ldots \times F_k$ with the metric $g = g_{F_0} \oplus u_1^2\, g_{F_1} \oplus \ldots \oplus u_k^2\, g_{F_k}$, where $u_i : F_0 \times F_i \to (0, \infty)$ for $1 \le i \le k$ are smooth functions.

Twisted products ($k = 1$) and **multiply warped products**, i.e., $u_i : F_0 \to (0, \infty)$, are special cases of multiply twisted products.

Let $\mathcal{D}_i$ be the involutive distribution on M corresponding to F_i. The **leaves** tangent to $\mathcal{D}_i$ $(i \ge 1)$ are totally umbilical submanifolds with the mean curvature vector fields $H_i = -n_i P_0 \nabla(\log u_i)$ tangent to $\mathcal{D}_0$, and the **fibers** (i.e., the leaves tangent to $\mathcal{D}_0$) are totally geodesic. We have (in terms of the warping functions u_i)

$$\mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} = \sum\nolimits_{i \ge 1} n_i\, (\Delta_{F_0}\, u_i)/u_i\,. \qquad (1)$$

Decomposition of the mixed scalar curvature

For the scalar curvature $\mathrm{S} : M \to \mathbb{R}$ of (M, g) we have

$$\mathrm{S} = 2\,\mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} + \sum\nolimits_{i=1}^{k} \mathrm{S}(\mathcal{D}_i),$$

where $\mathrm{S}(\mathcal{D}_i)$ – scalar curvatures of distributions (functions on M).

Proposition 1 ([2])

We have a useful decomposition formula for $\mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k}$*:*

$$2\,\mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} = \sum\nolimits_{i=1}^{k} \mathrm{S}_{\mathcal{D}_i,\mathcal{D}_i^\perp}. \tag{2}$$

Proof.

This follows from a simple formula for pairs $(\mathcal{D}_i, \mathcal{D}_i^\perp)$:

$$\mathrm{S}_{\mathcal{D}_i,\mathcal{D}_i^\perp} = \sum\nolimits_{j\ne i} \mathrm{S}_{\mathcal{D}_i,\mathcal{D}_j^\perp}$$

and the definition of $\mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k}$. $\square$

Extrinsic geometry of $(M, g; \mathcal{D}_1, \ldots, \mathcal{D}_k)$

Let $P_i : TM \to \mathcal{D}_i$ and $P_i^\perp : TM \to \mathcal{D}_i^\perp$ be orthoprojectors.

The 2nd fundamental form and integrability tensor of $\mathcal{D}_i$ are defined using the Levi-Civita connection ∇:

$$h_i(X,Y) = P_i^\perp(\nabla_X Y + \nabla_Y X)/2, \quad T_i(X,Y) = P_i^\perp\,[X,Y]/2, \quad X, Y \in \mathcal{D}_i.$$

The **mean curvature vector field** of $\mathcal{D}_i$ is $H_i = \operatorname{Tr}_g\, h_i$.

A distribution $\mathcal{D}_i$ is called **totally umbilical**, **harmonic**, or **totally geodesic**, if, respectively, $h_i = (H_i/n_i)\, g$, $\;H_i = 0$, $\;h_i = 0$.

The **shape operator** $(A_i)_Z$ of $\mathcal{D}_i$ with $Z \in \mathcal{D}_i^\perp$ and the operator $(T_i^\sharp)_Z$ are defined for $X, Y \in \mathcal{D}_i$ by

$$\langle (A_i)_Z(X), Y\rangle = \; h_i(X,Y), Z\rangle, \quad \langle (T_i^\sharp)_Z(X), Y\rangle = \langle T_i(X,Y), Z\rangle.$$

Similarly, $h_i^\perp$, $H_i^\perp = \operatorname{Tr}_g h_i^\perp$, $T_i^\perp$ are defined for $\mathcal{D}_i^\perp$,
h_{ij}, $H_{ij} = \operatorname{Tr}_g h_{ij}$, T_{ij} are defined for $\mathcal{D}_{ij} := \mathcal{D}_i \oplus \mathcal{D}_j$, etc.

Integral formulas for foliations are obtained by applying the Divergence Theorem to appropriate vector fields.

The following formula [6] plays a key role in my talk:

$$\underline{\operatorname{div}(H + H^\perp)} = \mathrm{S}_{\mathcal{D},\mathcal{D}^\perp} - Q(\mathcal{D}, g), \tag{3}$$

$$Q(\mathcal{D}, g) = \langle H^\perp, H^\perp\rangle + \langle H, H\rangle - \langle h, h\rangle - \langle h^\perp, h^\perp\rangle + \langle T, T\rangle + \langle T^\perp, T^\perp\rangle \tag{4}$$

Applying the Divergence Theorem to (3), we have

$$\int_M \big(\mathrm{S}_{\mathcal{D},\mathcal{D}^\perp} - Q(\mathcal{D}, g)\big)\, d\,\mathrm{vol}_g = 0. \tag{5}$$

Remark 1

For a codimension one foliation, (5) *simplifies to*

$$\int_M \big(\,\sigma_2 - \frac{1}{2}\,\mathrm{Ric}_{N,N}\big)\,\mathrm{d\,vol} = 0, \tag{6}$$

where $\sigma_2 = \sum_{i<j} k_i k_j$ is the second elementary symmetric function of the principal curvatures k_i of the leaves.

The following our formulas generalize (3) and (5) for $k > 2$.

Theorem 4 ([2])

For $(M, g; \mathcal{D}_1, \ldots, \mathcal{D}_k)$ **we have**

$$\operatorname{div} \sum\nolimits_i (H_i + H_i^\perp) = 2\,\mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} - \sum\nolimits_i Q(\mathcal{D}_i, g), \tag{7}$$

where $Q(\mathcal{D}_i, g)$ *are given in* (4) *with* $\mathcal{D} = \mathcal{D}_i$ *.*

For a closed manifold M**, by the Divergence Theorem, we get**

$$\int_M \big(\,2\,\mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} - \sum\nolimits_i Q(\mathcal{D}_i, g)\,\big)\,\mathrm{d}\,\mathrm{vol}_g = 0\,. \tag{8}$$

Example: n codimension-one foliations

Let (M^n, g) admits n pairwise orthogonal codimension-one foliations $\mathcal{F}_i$, and let N_i be unit vector fields orthogonal to $\mathcal{F}_i$. Writing down (6) for each N_i, summing for $i = 1, \ldots, n$, and using

$$\mathrm{S} = \sum\nolimits_i \mathrm{Ric}_{\,N_i,N_i},$$

yields the **integral formula** with the scalar curvature S of (M, g),

$$\int_M \big(\, 2 \sum\nolimits_i \sigma_2(\mathcal{F}_i) - \mathrm{S} \,\big)\, \mathrm{d}\,\mathrm{vol}_g = 0. \qquad (9)$$

Two immediately consequences of (9) (with $g > 0$):

- if $\mathrm{S} < 0$ then each foliation $\mathcal{F}_i$ cannot be totally umbilical;
- if $\mathrm{S} > 0$ then each $\mathcal{F}_i$ cannot be harmonic.

Splitting of $(M, g; \mathcal{D}_1, \ldots, \mathcal{D}_k)$

We say that $(M, g; \mathcal{D}_1, \ldots, \mathcal{D}_k)$ **splits** if all distributions $\mathcal{D}_i$ are integrable and M is locally the direct product $M_1 \times \ldots \times M_k$ with canonical foliations tangent to $\mathcal{D}_i$ and $g = g_1 \oplus \ldots \oplus g_k$.

If a simply connected manifold splits then it is the direct product.

Definition 5

A pair $(\mathcal{D}_i, \mathcal{D}_j)$ $(i \neq j)$ of distributions on $(M, g; \mathcal{D}_1, \ldots, \mathcal{D}_k)$ is

- **mixed totally geodesic**, if $h_{ij}(X, Y) = 0$ $(X \in \mathcal{D}_i, Y \in \mathcal{D}_j)$.
- **mixed integrable**, if $T_{ij}(X, Y) = 0$ $(X \in \mathcal{D}_i,\ Y \in \mathcal{D}_j)$.

Applications to splitting of manifolds

Theorem 6 ([2])

Let $(M, g; \mathcal{D}_1, \ldots, \mathcal{D}_k)$ with $g > 0$ has integrable harmonic distributions $\mathcal{D}_1, \ldots, \mathcal{D}_k$ such that each pair $(\mathcal{D}_i, \mathcal{D}_j)$ is mixed integrable. If $\underline{\mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} \ge 0}$, then (M, g) splits.

Theorem 7 ([2])

Let $(M, g; \mathcal{D}_1, \ldots, \mathcal{D}_k)$ with $g > 0$ has totally umbilical distributions such that each pair $(\mathcal{D}_i, \mathcal{D}_j)$ is mixed totally geodesic and $\langle H_i, H_j \rangle = 0$ for all $i \ne j$. If M is closed and $\underline{\mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} \le 0}$, then (M, g) splits.

Corollary 8 ([2])

Let a multiply twisted product manifold (M, g) satisfy $\langle H_i, H_j \rangle = 0$ for $i \ne j$. If M is closed and $\underline{\mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} \le 0}$, then (M, g) is the direct product.

Let M be a hypersurface in a Riemannian manifold $\bar{M}(c)$ of constant curvature c. M is called a **Dupin hypersurface** if

- the multiplicity of each principal curvature is constant on M;
- each principal curvature is constant along its principal directions.

It is known that **a compact Dupin hypersurface M in $\bar{M}(c)$ $(c \geq 0)$ has 1, 2, 3, 4 or 6 distinct principal curvatures**, that is $(M, g, \mathcal{D}_1, \ldots, \mathcal{D}_k)$ with $k = 2, 3, 4, 6$.

$A : TM \to TM$ the shape operator, X_i the principal directions on TM, and $AX_i = \mu_i X_i$. The sectional curvature of $M \subset \bar{M}(c)$ is

$$K(X_i, X_j) = c + \mu_i \, \mu_j, \quad i \neq j. \tag{10}$$

If M has two distinct principal curvatures $\mu_1 < \mu_2$, then using

$$\|h_i\|^2 - \|H_i\|^2 = \frac{n_i(1-n_i)\|\nabla\mu_i\|^2}{(\mu_i - \mu_j)^2}, \quad \mathrm{S}_{\mathcal{D}_1,\mathcal{D}_2} = n_1 n_2(c + \mu_1\mu_2),$$

integral formula (5) can be rewritten as, see [6]:

$$\int_M \Big(n_1 n_2(c + \mu_1\mu_2) + \frac{n_1(1-n_1)\,\|\nabla\mu_1\|^2}{(\mu_2-\mu_1)^2} + \frac{n_2(1-n_2)\,\|\nabla\mu_2\|^2}{(\mu_2-\mu_1)^2} \Big)\, d\,\mathrm{vol}_g = 0. \quad (11)$$

The next **problem** was posed by P. Walczak [6]: "To search for formulae analogous to (11) in the case of a hypersurface in $\bar{M}(c)$ of $k > 2$ distinct principal curvatures of constant multiplicities".

Application of (7) to Dupin hypersurfaces with $k = 3$

By (10), for $M \subset \bar{M}(c)$ with three distinct principal curvatures of constant multiplicities corresponding to distributions $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$:

$$\mathrm{S}_{\mathcal{D}_1,\mathcal{D}_2,\mathcal{D}_3} = \sum\nolimits_{i<j} n_i\, n_j (c + \mu_i \mu_j).$$

Theorem 9 ([2])

Let M be a hypersurface in $\bar{M}(c)$ with three distinct principal curvatures $\mu_1 < \mu_2 < \mu_3$ of constant multiplicities. Then

$$\operatorname{div} \sum\nolimits_{i=1}^{3} n_i \Big(\frac{P_j \nabla \mu_i}{\mu_i - \mu_j} + \frac{P_l \nabla \mu_i}{\mu_i - \mu_l}\Big) = \frac{1}{2} \sum\nolimits_{i<j} n_i\, n_j (c + \mu_i \mu_j)$$
$$+ \sum\nolimits_{i=1}^{3} n_i (1 - n_i) \Big(\frac{\|P_j \nabla \mu_i\|^2}{(\mu_i - \mu_j)^2} + \frac{\|P_l \nabla \mu_i\|^2}{(\mu_i - \mu_l)^2}\Big), \tag{12}$$

where $(j, l) \in \{1, 2, 3\} \setminus \{i\}$ and $j < l$. If M is closed, then ...

3. Canonical ("best") Riemannian metrics

Two general points of view on canonical Riemannian metrics:

1. **Curvature conditions**: we impose the constancy of Riem, or of its traces, namely, the Ricci and the scalar curvatures, and get space forms, Einstein manifolds and, Yamabe metrics.
Using the covariant derivative ∇, we get locally symmetric metrics, etc. Adding potential into consideration, we get Ricci solitons, etc.

2. **Critical metrics** of suitable Riemannian functionals are, by definition, tensorial solutions of the associated **Euler-Lagrange equations**: the Einstein-Hilbert action (most famous geometrically meaningful functional), also quadratic curvature functionals.

We can define and study new functionals that have suitable classical canonical metrics as a subset of their critical points.

The Einstein-Hilbert type action

An analog of the Einstein-Hilbert action, where the scalar curvature is replaced on $(M, g; \mathcal{D}_1, \ldots, \mathcal{D}_k)$ by $\mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k}$, is

the **mixed Einstein-Hilbert action**, see [3, 4],

$$J_{\mathcal{D}} : g \mapsto \int_M \Big\{ \frac{1}{2\mathfrak{a}} (\mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} - 2\,\Lambda) + \mathcal{L} \Big\}\, \mathrm{d}\,\mathrm{vol}_g \tag{13}$$

with a "cosmological constant" Λ, a Lagrangian $\mathcal{L}$ describing the "matter contents", and the coupling constant $\mathfrak{a} > 0$.

The geometrical part of (13) is the **total mixed scalar curvature**,

$$J^g_{\mathcal{D}} : g \mapsto \int_M \mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k}\, \mathrm{d}\,\mathrm{vol}_g\ . \tag{14}$$

To deal with non-compact manifolds one may integrate over arbitrarily large, relatively compact domain $\Omega \subset M$, which also contains supports of variations $B(t) := \partial g_t / \partial t$.

Adapted variations of metric

A family of metrics $g_t \in \mathrm{Riem}(M, \mathcal{D}_1, \ldots \mathcal{D}_k)$, $|t| < \varepsilon$, will be called an **adapted variation** if $\mathcal{D}_i$ and $\mathcal{D}_j$ are g_t-orthogonal for all $i \neq j$ and all time t.

An adapted variation g_t will be called a $\mathcal{D}_j$-**variation** if the metric g_t changes along $\mathcal{D}_j$ only.

An adapted variation g_t will be called **volume-preserving** if $\mathrm{Vol}(\Omega, g_t) = \mathrm{Vol}(\Omega, g)$ for all time t.

Figure: Variation of Salvador Dali

The Einstein type equation

Varying the action (13) in the class of **volume-preserving adapted** metrics, we obtain the **Euler-Lagrange equation** in the beautiful form of the Einstein equation

$$\mathcal{R}ic_{\mathcal{D}} - (1/2)\,\mathcal{S}_{\mathcal{D}} \cdot g + \Lambda\, g = \mathfrak{a} \cdot \Theta\,, \tag{15}$$

where the Ricci tensor and the scalar curvature are replaced by the Ricci type tensor $\mathcal{R}ic_{\mathcal{D}}$ and its trace $\mathcal{S}_{\mathcal{D}}$, which are defined later, and Θ is the "energy-momentum" tensor, given in a coordinates by $\Theta_{\mu\nu} = -2\,\partial\mathcal{L}/\partial g^{\mu\nu} + g_{\mu\nu}\mathcal{L}$.

In order to present $\mathcal{R}ic_{\mathcal{D}}$, define Casorati type operators $\mathcal{A}_i$ and $\mathcal{T}_i$, the $(0,2)$-tensor Ψ_i and a self-adjoint $(1,1)$-tensor $\mathcal{K}_i$, for $\mathcal{D}_i$:

$$\mathcal{A}_i = \sum\nolimits_{\,n_{\,i-1}<a\,\le n_i} \varepsilon_a (A_i)^2_{E_a}, \quad \mathcal{T}_i = \sum\nolimits_{\,n_{\,i-1}<a\,\le n_i} \varepsilon_a (T^\sharp_i)^2_{E_a},$$

$$\Psi_i(X,Y) = \operatorname{Tr}((A_i)_Y (A_i)_X + (T^\sharp_i)_Y (T^\sharp_i)_X), \quad X, Y \in \mathcal{D}_i^\perp,$$

$$\mathcal{K}_i = \sum\nolimits_{\,n_{\,i-1}<a\,\le n_i} \varepsilon_{\,a} \,[(T_i)_{E_a}, (A_i)_{E_a}].$$

Define also the $(0,2)$-tensor Υ_{P_1,P_2} implicitly by

$$\langle \Upsilon_{P_1,P_2}, S\rangle \ = \sum\nolimits_{\lambda,\mu} \varepsilon_\lambda\, \varepsilon_\mu \big[S(P_1(e_\lambda, e_\mu), P_2(e_\lambda, e_\mu)) + S(P_2(e_\lambda, e_\mu), P_1(e_\lambda, e_\mu)) \big],$$

for any $(1,2)$-tensors P_1, P_2 and a $(0,2)$-tensor S, where $\{e_\lambda\}$ is a local orthonormal basis of TM and $\varepsilon_\lambda = \langle e_\lambda, e_\lambda\rangle \in \{-1, 1\}$.

Variation formulas on $(M, g; \mathcal{D}, \mathcal{D}^\perp)$

In order to derive Euler-Lagrange equations for (14), we need variations of $\mathrm{S}_{\mathcal{D},\mathcal{D}^\perp}$, see (2), for which we use variations of six terms of $Q(\mathcal{D}, g)$ in (4), given in the following.

Proposition 2 ([5])

If g_t is a $\mathcal{D}$-variation of a metric $g \in \mathrm{Riem}(M, \mathcal{D}, \mathcal{D}^\perp)$, then

$$
\begin{aligned}
\partial_t \langle h^\perp, h^\perp\rangle &= -\langle (1/2)\Upsilon_{h^\perp, h^\perp},\ B\rangle,\\
\partial_t \langle h,\ h\rangle &= \langle \operatorname{div} h + \mathcal{K}^\flat,\ B\rangle - \underline{\operatorname{div}\langle h, B\rangle},\\
\partial_t \langle H^\perp, H^\perp\rangle &= -\langle H^\perp \otimes H^\perp,\ B\rangle,\\
\partial_t \langle H, H\rangle &= \langle (\operatorname{div} H)\, g,\ B\rangle - \underline{\operatorname{div}((\operatorname{Tr}_{\mathcal{D}} B^\sharp) H)},\\
\partial_t \langle T^\perp, T^\perp\rangle &= \langle (1/2)\Upsilon_{T^\perp, T^\perp},\ B\rangle,\\
\partial_t \langle T,\ T\rangle &= \langle 2\,\mathcal{T}^\flat,\ B\rangle.
\end{aligned}
\tag{16}
$$

Corollary 10 ([3])

For any $\mathcal{D}_j$-variation of metric $g \in \mathrm{Riem}(M, \mathcal{D}_1, \ldots, \mathcal{D}_k)$ we have

$$\partial_t \sum\nolimits_i Q(\mathcal{D}_i, g) = \langle \mathcal{Q}(\mathcal{D}_j, g),\ B_j\rangle - \underline{\operatorname{div} X_j}\,, \tag{17}$$

where $B_j = \partial_t\, g_{t\,|\,t=0}$ and
$2X_j = \langle h_j, B_j\rangle - (\operatorname{Tr}_{\mathcal{D}_j} B_j^\sharp)H_j + \sum_{i\neq j}(\langle h_i^\perp, B_j\rangle - (\operatorname{Tr}_{\mathcal{D}_i^\perp} B_j^\sharp)H_i^\perp)$.
Also

$$\mathcal{Q}(\mathcal{D}_j, g) = \sum\nolimits_i \mathcal{Q}_{j,i}, \quad 1 \le j \le k,$$

where the (0,2)-tensors $\mathcal{Q}_{j,i}$ are defined by (see Proposition 2)

$$\mathcal{Q}_{j,i} = \begin{cases} -\operatorname{div} h_j - \mathcal{K}_j^\flat - H_j^\perp \otimes H_j^\perp + (1/2)\Upsilon_{h_j^\perp, h_j^\perp} & \\ \quad +(1/2)\Upsilon_{T_j^\perp, T_j^\perp} + 2\,\mathcal{T}_j^\flat + (\operatorname{div} H_j)\, g_j, & i = j, \\ -\operatorname{div} h_i^\perp - (\mathcal{K}_i^\perp)^\flat - H_i \otimes H_i + (1/2)\Upsilon_{h_i, h_i} & \\ \quad +(1/2)\Upsilon_{T_i, T_i} + 2\,(\mathcal{T}_i^\perp)^\flat + (\operatorname{div} H_i^\perp)\, g_j, & i \neq j. \end{cases}$$

Euler-Lagrange equations for the action $J_{\mathcal{D}}$

The following theorem (based on Corollary 10 for divergence terms) allows us to build the partial Ricci curvature $\mathcal{R}ic_{\mathcal{D}}$.

Theorem 11 (Euler-Lagrange equations, [3])

A metric $g \in \mathrm{Riem}(M, \mathcal{D}_1, \ldots, \mathcal{D}_k)$ on M is critical for (14) with respect to volume-preserving adapted variations if and only if

$$\mathcal{Q}(\mathcal{D}_j, g) + \big(\,\mathrm{S}_{\,\mathcal{D}_1,\ldots,\mathcal{D}_k} - \frac{1}{2}\,\mathrm{div}\sum\nolimits_i (H_i + H_i^\perp) + \lambda_j\big)\, g_j = 0 \qquad (18)$$

for $1 \le j \le k$ and some $\lambda_j \in \mathbb{R}$.

Substituting the expression of $\mathcal{Q}(\mathcal{D}_j, g)$, equations (18) read as

$$\sum\nolimits_{i\ne j}\big(\,\mathrm{div}\, h_i^\perp + (\mathcal{K}_i^\perp)^\flat + H_i \otimes H_i - \frac{1}{2}\Upsilon_{h_i,h_i} - \frac{1}{2}\Upsilon_{T_i,T_i} - 2\,(\mathcal{T}_i^\perp)^\flat\big)$$
$$+\,\mathrm{div}\, h_j + \mathcal{K}_j^\flat - \frac{1}{2}\Upsilon_{h_j^\perp,h_j^\perp} + H_j^\perp \otimes H_j^\perp - \frac{1}{2}\Upsilon_{T_j^\perp,T_j^\perp} - 2\,\mathcal{T}_j^\flat$$
$$-\big(\,\mathrm{S}_{\,\mathcal{D}_1,\ldots,\mathcal{D}_k} - \mathrm{div}(H_j + \sum\nolimits_{i\ne j} H_i^\perp) + \lambda_j\big)\, g_j = 0, \quad 1 \le j \le k.$$

Using Theorem 11, we present $\mathcal{R}ic_{\mathcal{D}}$ explicitly.

Theorem 12 ([3])

A metric $g \in \mathrm{Riem}(M, \mathcal{D}_1, \ldots \mathcal{D}_k)$ is critical for the action (14) with respect to volume-preserving adapted variations if and only if g satisfies (15), where the Ricci type (symmetric) tensor $\mathcal{R}ic_{\mathcal{D}}$ is given by its restrictions on k subbundles $\mathcal{D}_j$ of TM,

$$\mathcal{R}ic_{\mathcal{D}\,|\,\mathcal{D}_j\times\mathcal{D}_j} = -\mathcal{Q}(\mathcal{D}_j, g) + \mu_j\, g_j, \quad j = 1, \ldots, k, \tag{19}$$

and μ_j are given in (21) below.

The values of μ_i

If $\mathcal{R}ic_{\mathcal{D}\,|\,\mathcal{D}_j\times\mathcal{D}_j}$ of (19) satisfy the **Einstein type equation** (15), then (μ_j) satisfy the linear system

$$\sum\nolimits_{i} n_i\,\mu_i - 2\,\mu_j = a_j, \quad j = 1,\ldots,k \tag{20}$$

with coefficients

$$a_j = \mathrm{Tr}_g \sum\nolimits_{i} \mathcal{Q}(\mathcal{D}_i, g) - 2\,\mathcal{Q}(\mathcal{D}_j, g).$$

The system (20) has a unique solution

$$\mu_{\,i} = -\frac{1}{2n-4}\,\big(\sum\nolimits_{j}(a_{\,i} - a_j)\,n_j - 2\,a_{\,i}\big), \quad 1 \le i \le k. \tag{21}$$

Future plans: singular distributions and extended gravity

- **Singular distributions**: the dimension of $\mathcal{D}_i(x)$ can depend on a point $x \in M$, see for $k = 2$:

[7] P. Popescu, V. Rovenski and S. Stepanov, **On singular distributions with statistical structure**, Mathematics, MDPI, 2020, 20 pp.

- Our action can be useful in studying the interaction of (singular) distributions, in "multi-time" geometric dynamics, developed in

[8] C. Udriște and I. Tevy, **Geometric Dynamics on Riemannian Manifolds**. Mathematics 8, 79 (2020).

- One can consider $(M, g, \bar\nabla = \nabla + \mathfrak{T}; \mathcal{D}_1, \ldots, \mathcal{D}_k)$, see [2], and study the following **perturbed Einstein-Hilbert-Cartan action**:

$$\bar J_\varepsilon : (g, \mathfrak{T}) \mapsto \int_M \{\frac{1}{2\mathfrak{a}}(\mathrm{S} + \varepsilon\,\mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} - 2\,\Lambda) + \mathcal{L}\}\,\mathrm{d}\,\mathrm{vol}_g, \quad \varepsilon \in \mathbb{R},$$

its critical points are space-times of "extended theory of gravity".

THANK  **YOU !**