$\boxed{\text{Floyd's manifold is a conjugation space}}$

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joint work with W. Pitsch

§1. Conjugation spaces (Hausmann, Holm, Puppe, 2005)

Let C_2 be a cyclic group of order. We work with connected spaces / manifolds with a C_2-action.

Consider spaces with $H^*(X;\mathbb{F}_2)$ concentrated in even degrees together with an isomorphism of graded $\mathbb{F}_2$-vector spaces

$$\kappa : H^{2*}(X;\mathbb{F}_2) \longrightarrow H^*(X^{C_2};\mathbb{F}_2)$$

Borel construction $X_{hC_2} = X \times EC_2 / C_2$

Borel cohomology $H^*_{C_2}(X;\mathbb{F}_2) = H^*(X_{hC_2};\mathbb{F}_2)$

Fibration $X \xrightarrow{i} X_{hC_2} \twoheadrightarrow *_{hC_2} = BC_2 = \mathbb{R}P^\infty$

induces $H^*_{C_2}(X;\mathbb{F}_2) \xrightarrow{i^*} H^*(X;\mathbb{F}_2)$

Require i^* to have a section $\sigma : H^*(X;\mathbb{F}_2) \to H^*_{C_2}(X;\mathbb{F}_2)$

Def: X is a <u>conjugation space</u> if it has a couple $(\varkappa, \sigma)$ st. the <u>conjugation equation</u> holds:

$$\begin{array}{l} H^{2n}(X;\mathbb{F}_2) \xrightarrow{\sigma} H^{2n}(X \times_{C_2} EC_2;\mathbb{F}_2) \longrightarrow H^{2n}(\underbrace{X^{C_2} \times_{C_2} EC_2}_{X^{C_2} \times BC_2};\mathbb{F}_2) \\ \\ x \longmapsto \varkappa(x)\cdot u^n + \underbrace{lt_n}_{\text{lower terms degree } <n \text{ in } u} \quad \big(\underbrace{H^*(X^{C_2};\mathbb{F}_2)[u]}_{\text{degree } 1}\big)^{2n} \cong H^{2n}(X^{C_2} \times BC_2;\mathbb{F}_2) \end{array}$$

<u>Example:</u>

- $(\mathbb{C}^n)^c = S^{2n} = S^{n\rho} = S^{n(1+\sigma)}$
  $\rho = 1+\sigma$ regular representation
  $x \longmapsto \varkappa(x)\cdot u^n$
- $\mathbb{C}P^n$, $x \longmapsto \varkappa(x)u^n + Sq^1\varkappa(x)u^{n-1} + \dots + Sq^n\varkappa(x)$
- cx Grassmannians
- spherical spaces built from complex conjugation cells, unit balls in $\mathbb{C}^n$ & eq. attaching maps.

Consequences

- $(\varkappa, \sigma)$ is unique
- $\varkappa$ is an isomorphism of $\mathbb{F}_2$-algebras
- Franz-Puppe: $\varkappa(Sq^{2k} x) = Sq^k \varkappa(x)$

Another viewpoint (j. w. W. Pitsch, N. Ricka)

non-eq: $X \wedge H\mathbb{F}_2$ splits as $\bigvee M(H_n(X;\mathbb{F}_2), n) \wedge H\mathbb{F}_2$ (Moore spaces)

eq: we use $H\underline{\mathbb{F}_2}$, $\underline{\mathbb{F}_2}$ Mackey functor $\circlearrowright\left(\begin{smallmatrix}\mathbb{F}_2\\ \mathbb{F}_2\end{smallmatrix}\right) id$

Theorem (Pitsch-Ricka-S)
Let X be a compact C_2-space. Then X is a conj. space iff it is pure, ie $X \wedge H\underline{\mathbb{F}_2} \simeq \bigvee_i S^{n_i \rho} \wedge H\underline{\mathbb{F}_2}$

Ex: $\mathbb{C}P^n \wedge H\underline{\mathbb{F}_2} \simeq \bigvee_{k=0}^{n} S^{k\rho} \wedge H\mathbb{F}_2$

C. May: X compact C_2-space, $X \wedge H\underline{\mathbb{F}_2}$ always splits.

§2 Floyd manifolds (1973)

Adams: $X = e^0 \cup e^n \cup e^{2n}$

Floyd: $X = e^0 \cup e^k \cup e^{n+k} \cup e^n$

There are only two non-trivial cobordism classes of such manifolds, namely in dimensions 5 and 10. Call them F_5 and F_{10}. Start with $\overline{F}_{10} = S^3 \times_{S^1} \mathbb{H}P^2$

$S^3 \subset \underset{S^1}{\underset{\cup}{\mathbb{C}}} \times \underset{S^1}{\underset{\cup}{\mathbb{C}}}$, S^1 acts diagonally, C_2 acts by conj. on each copy of $\mathbb{C}$

$1, i \qquad j, k$

$\mathbb{H}P^2$ built from $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$,

S^1 acts only on $\mathbb{C}j$, C_2 acts by conj. on both copies.

There is a fibration $\mathbb{H}P^2 \longrightarrow \overline{F}_{10} \underset{\curvearrowleft s}{\longrightarrow} S^3 \times_{S^1} * = S^2$

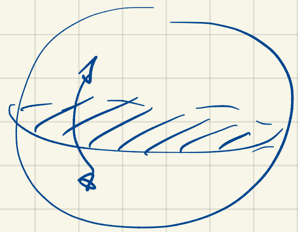
$\overline{F}_{10}$ is a conjugation space, $\overline{F}_{10}^{C_2} = \overline{F}_5 = S^1 \times_{C_2} \mathbb{C}P^2$

$\tilde{H}^* \overline{F}_{10}$:

10	8	6	4	2
$\bullet$	$\cancel{\bullet}$	$\bullet$	$\bullet$	$\cancel{\bullet}$

Consider a tubular neigh of $s(S^2) \subset \overline{F}_{10}$ and do equivariant surgery.

Lück–Uribe, 2014. Consider the $GL_8(\mathbb{R})$-bundle associated to s, over S^5 $\rightsquigarrow$ $f : S^5 \longrightarrow B(C_2; GL_8(\mathbb{R}))$

Wish $f \simeq *$: free pointed C_2-space $S^2 \vee S^2 \overset{C_2}{\simeq}$  $\xrightarrow{\bar{f}} B(C_2; GL_8(\mathbb{R}))$

S^5

$$f^{C_2} : S^1 \longrightarrow B(C_2; GL_8(\mathbb{R}))^{C_2} \simeq B(GL_4(\mathbb{R}) \times GL_4(\mathbb{R}))$$

Floyd: $[f^{C_2}] = *$.

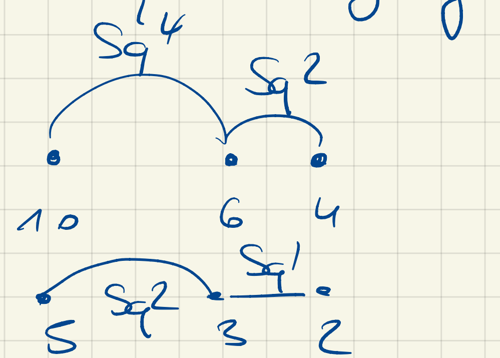
free-forget: $S^2 \longrightarrow B(C_2; GL_8(\mathbb{R}))^e = BGL_8(\mathbb{R})$

non-eq. version of $f \simeq *$ (Floyd).

$\Rightarrow$ $f \simeq_{C_2} *$: can perform C_2-eq. surgery

Get $F_{10} = S^4 \cup e^6 \cup e^{10}$

$(F_{10})^{C_2} = F_5$



$\underline{\text{Theorem}}$ (Pitsch S)

F_{10} is a conjugation space

$\underline{\text{proof}}$: $F_{10} \wedge H\underline{\mathbb{F}_2}$ splits (C. Ney)

identify all summands $S^{2\rho} \wedge H\underline{\mathbb{F}_2} \vee S^{3\rho} \wedge H\underline{\mathbb{F}_2}$

$\vee S^{5\rho} \wedge H\underline{\mathbb{F}_2}$ $\square$

$\underline{\text{Note}}$: F_{10} is the smallest space we know which cannot be realized as X^{C_2} for X conj. space.