Equivariant Poincaré duality for finite groups and fixed points methods

joint work–in–progress with Dominik Kirstein & Christian Kremer

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University of Haifa Topology & Geometry Seminar

9th April 2024

1. Background
2. Nonequivariant theory review
3. Equivariant theory
4. Application: theorem of the single fixed point

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Classical statement

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Let X be an orientable compact space. Then there is an integer n and an isomorphism of graded groups

$$H_*(X;\mathbb{Z}) \cong H^{n-*}(X;\mathbb{Z})$$

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- **Theoretical:** starting point for surgery theory

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in terms of matching Betti numbers

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2023 **B. Cnossen** (PhD thesis)
studied a "pre–equivariant Poincaré duality" situation he called twisted ambidexterity

Goal of project

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To develop a theory of equivariant Poincaré duality for finite groups to the extent of being able to relate and exploit the relationships between the different fixed points in nontrivial ways.

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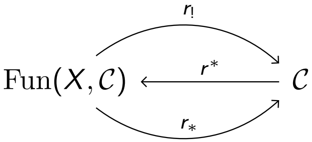

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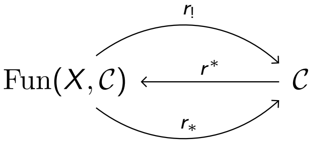

For $\zeta \in \mathrm{Fun}(X, \mathcal{D}(\mathbb{Z}))$, we have

$$\pi_*(r_!\zeta) \cong H_*(X;\zeta) \quad \text{and} \quad \pi_{-*}(r_*\zeta) \cong H^*(X;\zeta)$$

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Remark: In fact, more generally, when $\mathcal{C}$ is a presentably symmetric monoidal stable category, it is just a property for a compact space to be $\mathcal{C}$–Poincaré duality.

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$$\begin{array}{ccc} \mathcal{C}^H & \xrightarrow{\varphi_H} & \mathcal{D}^H \\ {\scriptstyle \mathrm{Res}_K^H}\downarrow & \equiv & \downarrow{\scriptstyle \mathrm{Res}_K^H} \\ \mathcal{C}^K & \xrightarrow{\varphi_K} & \mathcal{D}^K \end{array}$$

for every subgroup inclusion $K \le H \le G$.

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Examples: Smooth G–manifolds and tom Dieck's generalised homotopy representations are $\underline{\mathrm{Sp}}$–Poincaré duality.

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Manoeuvre 2 (Poincaré isotropy): Let $\underline{X}$ be a compact G–space and $\mathcal{C}$ a presentably symmetric monoidal stable category. Then $\underline{X}$ is $s_*\mathcal{C}$–Poincaré duality if and only if X^G is $\mathcal{C}$–Poincaré duality.

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Corollary: If $\underline{X}$ is $\underline{\mathrm{Sp}}_G$–Poincaré duality, then X^G is Sp–Poincaré duality.

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Corollary: If $\underline{X}$ is $\underline{\mathrm{Sp}}_G$–Poincaré duality, then X^G is Sp–Poincaré duality. This is obtained by upgrading the geometric fixed points functor $\Phi^G\colon \mathrm{Sp}_G \to \mathrm{Sp}$ to a G–exact symmetric monoidal functor $\Phi\colon \underline{\mathrm{Sp}} \to s_*\mathrm{Sp}$.

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A theorem of Atiyah–Bott and Conner–Floyd

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Remark: proof is purely homotopy–theoretic using the theory of fundamental classes developed in the project.

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$$\left(r_!^{>1}\epsilon^*\zeta\right)^{tG} \xrightarrow{\simeq} \left(r_!\zeta\right)^{tG}$$

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- By a totally general argument, this map is nullhomotopic,
- By a group homology calculation, this map is π_0–surjective onto a nonzero group if $X^G \simeq *$. □

Thank You!