Modules over some ring spectra realising modules over subHopf algebras of the Steenrod algebra

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Recollections on mod 2 (co)homology

Each of $H_*(-) = H_*(-;\mathbb{F}_2)$ and $H^*(-) = H^*(-;\mathbb{F}_2)$ is a homotopy functor from spaces to $\mathbb{Z}$-graded vector spaces. The reduced theories $\widetilde{H}_*(-)$ and $\widetilde{H}^*(-)$ gives functors from based spaces to $\mathbb{Z}$-graded vector spaces which extend to spectra. A *stable cohomology operation* θ *of degree* k is a sequence of natural transformations

$$\theta_n\colon H^n(-) \to H^{n+k}(-) \quad (n \in \mathbb{Z})$$

compatible with suspension isomorphisms, i.e., the following diagram commutes for all n and k.

$$\begin{tikzcd}
\widetilde{H}^n(-) \arrow[r, "\theta_n"] \arrow[d, "\cong"'] & \widetilde{H}^{n+k}(-) \arrow[d, "\cong"] \\
\widetilde{H}^{n+1}(\Sigma(-)) \arrow[r, "\theta_{n+1}"] & \widetilde{H}^{n+k+1}(\Sigma(-))
\end{tikzcd}$$

The set of all such operations $\mathcal{A}^k = H^k(H)$ is an $\mathbb{F}_2$-vector space, and these form the mod 2 *Steenrod algebra* $\mathcal{A} = \mathcal{A}^* = H^*(H)$, a non-commutative graded algebra with composition as product. The structure of $\mathcal{A}$ was determined by Serre, then Milnor showed that it was a cocommutative Hopf algebra and determined its dual Hopf algebra $\mathcal{A}_*$ where $\mathcal{A}_n = \mathrm{Hom}_{\mathbb{F}_2}(\mathcal{A}^n, \mathbb{F}_2)$. As an algebra, $\mathcal{A}$ is generated by the *Steenrod operations* $\mathrm{Sq}^n \in \mathcal{A}^n$ ($n \geqslant 1$) satisfying the *Adem relations* (here $\mathrm{Sq}^0 = 1$):

$$\text{For } 0 < r < 2s, \quad \mathrm{Sq}^r \mathrm{Sq}^s = \sum_{0 \leqslant j \leqslant \lfloor r/2 \rfloor} \binom{s-1-j}{r-2j} \mathrm{Sq}^{r+s-j} \mathrm{Sq}^j .$$

There is a basis of *admissible monomials*

$$\mathrm{Sq}^{(i_1,\ldots,i_\ell)} = \mathrm{Sq}^{i_1} \mathrm{Sq}^{i_2} \cdots \mathrm{Sq}^{i_\ell}$$

where $i_{r-1} \geqslant 2i_r$ for $2 \leqslant r \leqslant \ell$ and $i_\ell \geqslant 1$. Here ℓ is the *length* of the monomial and the identity operation $\mathrm{Sq}^0 = 1$ is the only element of length zero.

In fact the Sq^{2^s} are the only algebra indecomposables, so $\mathcal{A}$ is generated as an algebra by these.
The cocommutative coproduct $\psi\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ and the antipode $\chi\colon \mathcal{A} \to \mathcal{A}$ are given by the formulae

$$\psi(\mathrm{Sq}^n) = \sum_{0\leqslant r\leqslant n} \mathrm{Sq}^r \otimes \mathrm{Sq}^{n-r}, \quad \sum_{0\leqslant r\leqslant n} \chi(\mathrm{Sq}^r)\,\mathrm{Sq}^{n-r} = 0.$$

The antipode is anti-commutative, i.e.,

$$\chi(\alpha\beta) = \chi(\beta)\chi(\alpha).$$

Here are the first few $\chi(\mathrm{Sq}^{2^s})$:

$$\chi(\mathrm{Sq}^1) = \mathrm{Sq}^1,\ \chi(\mathrm{Sq}^2) = \mathrm{Sq}^2,\ \chi(\mathrm{Sq}^4) = \mathrm{Sq}^4 + \mathrm{Sq}^1\,\mathrm{Sq}^4\,\mathrm{Sq}^1\,.$$

Theorem (Serre & Milnor)

The commutative Hopf algebra $\mathcal{A}_*$ is polynomial:

$$\mathcal{A}_* = \mathbb{F}_2[\xi_r : r \geqslant 1] = \mathbb{F}_2[\zeta_r : r \geqslant 1],$$

where $\xi_r, \zeta_r \in \mathcal{A}_{2^r-1}$ and $\zeta_r = \chi(\xi_r)$. The coproduct and antipode satisfy

$$\psi(\xi_n) = \sum_{0 \leqslant j \leqslant n} \xi_{n-j}^{2^j} \otimes \xi_j, \quad \psi(\zeta_n) = \sum_{0 \leqslant j \leqslant n} \zeta_j \otimes \zeta_{n-j}^{2^j},$$

$$\zeta_n = \sum_{1 \leqslant k \leqslant n} \xi_k \zeta_{n-k}^{2^k}.$$

The non-zero primitives are the elements $\xi_1^{2^s} = \zeta_1^{2^s}$.

The Poincaré series for $\mathcal{A}$ and $\mathcal{A}_*$ is $\displaystyle\prod_{r \geqslant 1} (1 - t^{2^r-1})^{-1}$.

Modules over $\mathcal{A}$

Modules will be cohomologically graded and assumed to be left modules. The dual of a module $M = M^*$ is $\mathrm{D}M = \mathrm{D}M^*$ where

$$\mathrm{D}M^n = \mathrm{Hom}_{\mathbb{F}_2}(M^{-n}, \mathbb{F}_2)$$

and its left action is given by

$$(\theta f)(-) = f(\chi\theta(-)).$$

Of course for a finite type spectrum X,

$$H_n(X) = \mathrm{Hom}_{\mathbb{F}_2}(H^n(X), \mathbb{F}_2) = (\mathrm{D}H^*(X))^{-n}.$$

If X is a finite CW spectrum with Spanier-Whitehead dual DX,

$$H^*(DX) \cong \mathrm{D}H^*(X).$$

Finite sub-Hopf algebras of $\mathcal{A}$

Important fact: $\mathcal{A} = \bigcup_{n\geqslant 0} \mathcal{A}(n)$, where $\mathcal{A}(n) \subseteq \mathcal{A}$ is the finite sub-Hopf algebra of dimension $2^{\binom{n+2}{2}}$ generated by $\mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^4, \ldots, \mathrm{Sq}^{2^n}$ with dual quotient Hopf algebra

$$\begin{aligned}\mathcal{A}(n)_* &= \mathcal{A}_*/(\zeta_1^{2^{n+1}}, \zeta_2^{2^n}, \zeta_3^{2^{n-1}}, \ldots \zeta_{n+1}^2, \zeta_{n+2}, \ldots) \\ &= \mathcal{A}_*//\mathbb{F}_2[\zeta_1^{2^{n+1}}, \zeta_2^{2^n}, \zeta_3^{2^{n-1}}, \ldots \zeta_{n+1}^2, \zeta_{n+2}, \ldots].\end{aligned}$$

Here $\mathcal{A}(n)$ and $\mathcal{A}(n)_*$ have Poincaré series

$$\prod_{1\leqslant r\leqslant n+1} \frac{(1 - t^{2^{n+2-r}(2^r-1)})}{(1 - t^{2^r-1})}.$$

The highest degree element in $\mathcal{A}(n)_*$ is the residue class of

$$z_n = \zeta_1^{2^{n+1}-1}\zeta_2^{2^n-1}\zeta_3^{2^{n-1}-1}\cdots\zeta_{n+1}$$

and dual to this is a generator of the top degree of $\mathcal{A}(n)$. The dual pairing makes z_n a Frobenius form so $\mathcal{A}(n)$ is a *Poincaré duality algebra* and thus self-injective.

The Wall relations

The Adem relations are neither minimal nor do they restrict to the $\mathcal{A}(n)$ subalgebras: for example, the identities

$$\mathrm{Sq}^2\,\mathrm{Sq}^3 = \mathrm{Sq}^4\,\mathrm{Sq}^1 + \mathrm{Sq}^5 = \mathrm{Sq}^4\,\mathrm{Sq}^1 + \mathrm{Sq}^1\,\mathrm{Sq}^4$$

are not meaningful in $\mathcal{A}(1)$ since $\mathrm{Sq}^4 \notin \mathcal{A}(1)$. Wall's relations amongst the generators Sq^{2^s} are minimal and restrict to give minimal relations for the $\mathcal{A}(n)$ subalgebras.

For $0 \leqslant s \leqslant r-2$ and $1 \leqslant t$, let

$$\Theta(r,s) = \mathrm{Sq}^{2^r}\,\mathrm{Sq}^{2^s} + \mathrm{Sq}^{2^s}\,\mathrm{Sq}^{2^r},$$

$$\Phi(t) = \mathrm{Sq}^{2^t}\,\mathrm{Sq}^{2^t} + \mathrm{Sq}^{2^{t-1}}\,\mathrm{Sq}^{2^t}\,\mathrm{Sq}^{2^{t-1}} + \mathrm{Sq}^{2^{t-1}}\,\mathrm{Sq}^{2^{t-1}}\,\mathrm{Sq}^{2^t}.$$

Then $\Theta(r,s) \in \mathcal{A}(r-1)$ and $\Phi(r) \in \mathcal{A}(r-1)$ so these can be expressed as polynomial expressions in the Sq^{2^k} for $0 \leqslant k \leqslant r-1$.

The elements

$$\mathrm{Sq}^{2^r}\mathrm{Sq}^{2^s} + \mathrm{Sq}^{2^s}\mathrm{Sq}^{2^r} + \Theta(r,s),$$
$$\mathrm{Sq}^{2^t}\mathrm{Sq}^{2^t} + \mathrm{Sq}^{2^{t-1}}\mathrm{Sq}^{2^t}\mathrm{Sq}^{2^{t-1}} + \mathrm{Sq}^{2^{t-1}}\mathrm{Sq}^{2^{t-1}}\mathrm{Sq}^{2^t} + \Phi(t)$$

give a minimal set of relations for $\mathcal{A}$. In particular, such elements with $r, t \leqslant n$ form a minimal set of relations for $\mathcal{A}(n)$.
In the first few cases the Wall relations are

$$\begin{aligned}
\mathcal{A}(0):\quad & \mathrm{Sq}^1\mathrm{Sq}^1 = 0,\\
\mathcal{A}(1):\quad & \mathrm{Sq}^1\mathrm{Sq}^1 = \mathrm{Sq}^2\mathrm{Sq}^2 + \mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^1 = 0\\
\mathcal{A}(2):\quad & \mathrm{Sq}^1\mathrm{Sq}^1 = \mathrm{Sq}^2\mathrm{Sq}^2 + \mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^1\\
& \qquad\qquad = \mathrm{Sq}^4\mathrm{Sq}^4 + \mathrm{Sq}^2\mathrm{Sq}^4\mathrm{Sq}^2 + \mathrm{Sq}^2\mathrm{Sq}^2\mathrm{Sq}^4\\
& \qquad\qquad = \mathrm{Sq}^1\mathrm{Sq}^4 + \mathrm{Sq}^4\mathrm{Sq}^1 + \mathrm{Sq}^2\mathrm{Sq}^1\mathrm{Sq}^2 = 0.
\end{aligned}$$

Using these it is possible to produce explicit bases for the $\mathcal{A}(n)$s. Here are the top dimensional elements in $\mathcal{A}(n)$ when $n = 0, 1, 2$:

$$\mathrm{Sq}^1,\ \mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^1\mathrm{Sq}^2,\ \mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^4.$$

Generalisations of the Steenrod algebra

Modern categories of spectra are symmetric monoidal with respect to smash products before passage to homotopy. The category of S-modules $\mathscr{M}_S$ is the earliest example and provides a good model for the category of spectra. In this category a commutative monoid is equivalent to an $\mathcal{E}_\infty$ ring spectrum and is called a *commutative S-algebra*. Examples include S, $H\mathbb{Z}$, $H\mathbb{F}_p$, $k\mathrm{O}$, $k\mathrm{U}$, $M\mathrm{U}$.

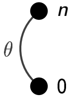
Every commutative S-algebra R has an associated category of modules $\mathscr{M}_R$ which is also closed symmetric monoidal with respect to a relative smash product $\wedge_R$ and function object $F_R(-,-)$; it also has a model structure and homotopy category $\mathscr{D}_R$ in which to do homotopy theory.

If R is connective and $\pi_0 R = \mathbb{Z}$ or $\pi_0 R = \mathbb{Z}_{(p)}$ there is a morphism of commutative S-algebras $R \to H = H\mathbb{F}_p$ so H is a commutative R-algebra, and then there are relative homology and cohomology theories

$$H^R_*(-) = \pi_*(H \wedge_R -), \quad H_R^*(-) = \pi_{-*}(F_R(-, H)).$$

The relative Steenrod algebra $H_R^*(H)$ is the algebra of stable operations in $H_R^*(-)$. If $R = S$ and $p = 2$, $H_S^*(H) = H^*(H) = \mathcal{A}$. Other examples: $H_{k\mathrm{O}}^*(H) = \mathcal{A}(1)$ and $H_{\mathrm{tmf}}^*(H) = \mathcal{A}(2)$.

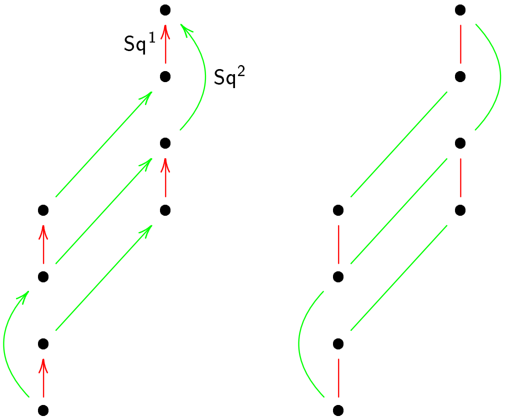
Realisation question: When working with spectra (or equivalently S-modules) we can ask whether an $\mathcal{A}$-module M is realisable as $H^*(X)$ for some S-module X. Similarly, for an $\mathcal{A}(1)$-module we can ask if it is $H_{k\mathrm{O}}^*(Y)$ for a $k\mathrm{O}$-module Y and for an $\mathcal{A}(2)$-module we can ask if it is $H_{\mathrm{tmf}}^*(Z)$ for a tmf-module Z.
Example: When can we realise an $\mathcal{A}$-module of the following form with $0 \neq \theta \in \mathcal{A}$?



Algebraic observation: Module only exists if θ is indecomposable, i.e., $n = 2^s$ and $\theta = \mathrm{Sq}^{2^s}$ +decomposables.
Hopf invariant 1 Theorem (Adams): Only realisable if $s = 0, 1, 2, 3$.

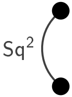
Realisability of $\mathcal{A}(1)$-modules

We will work with left modules $M = M^*$ involving multiplication maps $\mathcal{A}(1)^r \otimes M^n \to M^{n+r}$. Here some pictures of $\mathcal{A}(1)$ which is a free cyclic module realisable as $H^*_{k\mathrm{O}}(H)$.

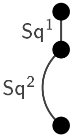


Here are some more realisable $\mathcal{A}(1)$-modules. In each case we can form a finite CW spectrum W then take $k\mathrm{O} \wedge W$ to get $H^*_{k\mathrm{O}}(k\mathrm{O} \wedge W) \cong H^*(W)$ with its $\mathcal{A}$-action restricted to an action of the subalgebra $\mathcal{A}(1) \subseteq \mathcal{A}$.

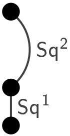
$\mathbb{F}_2 = H^*(S^0)$



$\mathcal{A}(1)/\mathcal{A}(1)\{\mathrm{Sq}^1, \mathrm{Sq}^1\,\mathrm{Sq}^2\} = H^*(C_\eta)$

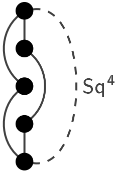


$\mathcal{A}(1)/\mathcal{A}(1)\{\mathrm{Sq}^1, \mathrm{Sq}^2\,\mathrm{Sq}^1\,\mathrm{Sq}^2\}$



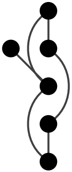
$\mathcal{A}(1)/\mathcal{A}(1)\{\mathrm{Sq}^2\}$

The Joker



$\mathcal{A}(1)/\mathcal{A}(1)\{\mathrm{Sq}^1\,\mathrm{Sq}^2\}$

The whiskered Joker

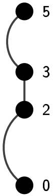


$\mathcal{A}(1)/\mathcal{A}(1)\{\mathrm{Sq}^2\,\mathrm{Sq}^1\,\mathrm{Sq}^2\}$

The construction of the Joker example uses the Toda bracket $\langle 2, \eta, 2\rangle = \{\eta^2\} \subseteq \pi_2(S^0)$. Later we'll see other examples of Toda brackets playing a rôle.

There are two different $\mathcal{A}$-module extensions of the Joker module. These differ in the action of Sq^4; the corresponding $\mathcal{A}$-modules are dual to each other. Their realisations are Spanier-Whitehead dual to each other but not weakly equivalent.

What about this one?



$\mathcal{A}(1)/\mathcal{A}(1)\{\mathrm{Sq}^1\}$

Let's first think about whether the above diagram can be realised as an $\mathcal{A}$-module. Notice that the top class is $\mathrm{Sq}^2\,\mathrm{Sq}^1\,\mathrm{Sq}^2$. Using Adem relations we have

$$\mathrm{Sq}^2\,\mathrm{Sq}^1\,\mathrm{Sq}^2 = \mathrm{Sq}^2\,\mathrm{Sq}^3 = \mathrm{Sq}^5 + \mathrm{Sq}^4\,\mathrm{Sq}^1 = \mathrm{Sq}^1\,\mathrm{Sq}^4 + \mathrm{Sq}^4\,\mathrm{Sq}^1$$

which is not possible. Despite this, there is a $k\mathrm{O}$-module realising this module, namely $H\mathbb{Z}$ for which $H^*_{k\mathrm{O}}(H\mathbb{Z}) \cong \mathcal{A}(1)/\mathcal{A}(1)\{\mathrm{Sq}^1\}$.

Another approach using a Toda bracket

The existence of a CW spectrum $W = S^0 \cup e^2 \cup e^3 \cup e^5$ whose cohomology is $\mathcal{A}/\mathcal{A}\{\mathrm{Sq}^1\}$ is equivalent to 0 being an element of the Toda bracket $\langle \eta, 2, \eta \rangle = \{\pm 2\nu\} \subseteq \pi_3(S^0)$. But $0 \notin \{\pm 2\nu\}$.

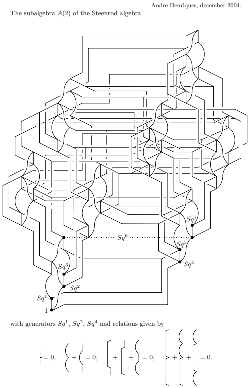
We can also interpret the Toda bracket as defined in $\pi_3(k\mathrm{O})$. Here the image of ν is 0, so we can build a CW $k\mathrm{O}$-module of this form; the result is equivalent to $H\mathbb{Z}$ as a $k\mathrm{O}$-module.

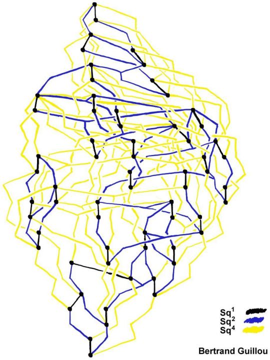
Here a $k\mathrm{O}$ cell is attached using a $k\mathrm{O}$-module map $S^{n-1}_{k\mathrm{O}} = k\mathrm{O} \wedge S^{n-1} \to X$ by forming its mapping cone $X \cup D^{n-1}_{k\mathrm{O}}$.

There are many other examples of realisable cyclic $\mathcal{A}(1)$-modules!

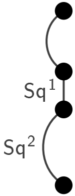
Realisability of $\mathcal{A}(2)$-modules with tmf-modules

Here are some pictures of $\mathcal{A}(2)$ (remember that $\dim \mathcal{A}(2) = 64$).





All of the examples for $k\mathrm{O}$ of the form $k\mathrm{O} \wedge W$ can be replaced by $\mathrm{tmf} \wedge W$ so that $H^*_{\mathrm{tmf}}(\mathrm{tmf} \wedge W) \cong H^*(W)$ as $\mathcal{A}(2)$-modules. The Sq^4 argument works to show there is no $\mathcal{A}(2)$-module



and the Toda bracket argument also applies since the image of ν in $\pi_3(\mathrm{tmf})$ has order 8.

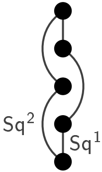
Many interesting $\mathcal{A}(2)$-modules can be obtained using *doubling* which exploits the fact that there is a degree halving surjective homomorphism of Hopf algebras $\mathcal{A}(2) \twoheadrightarrow \mathcal{A}(1)$ under which

$$\mathrm{Sq}^n \mapsto \begin{cases} \mathrm{Sq}^{n/2} & \text{if } n \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

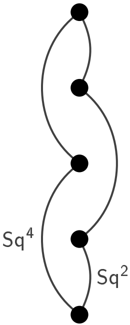
By restricting and doubling degrees, every $\mathcal{A}(1)$-module M induces an $\mathcal{A}(2)$-module ${}^{(1)}M$.

Doubling the Joker

Joker



${}^{(1)}$Joker

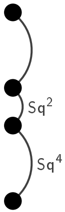


The following examples are of the form $H^*(W)$. Their constructions depending on $\eta\nu \in \pi_4(S^0) = 0$. The two CW spectra are stably Spanier-Whitehead dual.

$\mathcal{A}(2)/\mathcal{A}(2)\{\mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^7\}$ $\mathcal{A}(2)/\mathcal{A}(2)\{\mathrm{Sq}^1, \mathrm{Sq}^1\,\mathrm{Sq}^2, \mathrm{Sq}^4\}$

It is also possible to realise the double of the (whiskered) Joker using the Toda bracket $\langle \eta, \nu, \eta\rangle = \{\nu^2\} \subseteq \pi_6(S^0)$. The double of $\mathcal{A}(1)$ is also realisable as a spectrum so we can smash it with tmf to realise this $\mathcal{A}(2)$-module.

What about this one?



$\mathcal{A}(2)/\mathcal{A}(2)\{\mathsf{Sq}^1, \mathsf{Sq}^2\}$

We can't rule this out with Steenrod operations. What about a Toda bracket argument? Constructing a suitable CW complex requires the Toda bracket $\langle \nu, \eta, \nu \rangle \subseteq \pi_8(S^0)$ to contain 0. But

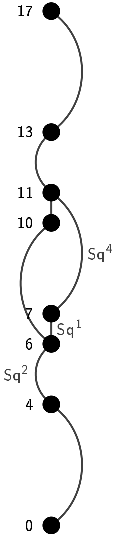
$$\langle \nu, \eta, \nu \rangle = \{\overline{\nu}\} = \{\eta\sigma + \varepsilon\} \not\ni 0.$$

Here the image of σ in $\pi_7(\mathrm{tmf})$ is 0 but the image of ε is not. This means that there is no tmf-module with this cohomology! If it did exist its homotopy would be $\pi_*(k\mathrm{O})[v_2]$.

Some tmf-modules related to $k\mathrm{O}$

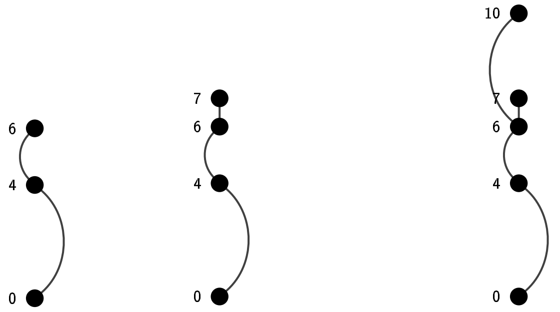
The cohomology of the tmf-module $k\mathrm{O}$ is shown below.

$$H^*_{\mathrm{tmf}}(k\mathrm{O}) \cong \mathcal{A}(2)/\mathcal{A}(2)\{\mathrm{Sq}^1, \mathrm{Sq}^2\}$$



We can realise $k\mathrm{O}$ as a CW tmf-module with cells corresponding to the basis shown. Each skeleton is a tmf-module whose cohomology is the cyclic $\mathcal{A}(2)$-module beneath it.

$k\mathrm{O}^{[6]}$ $\quad$ $k\mathrm{O}^{[7]}$ $\quad$ $k\mathrm{O}^{[10]}$

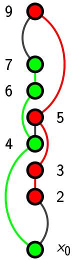


$\mathcal{A}(2)/\mathcal{A}(2)\{\mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^4\}$ $\quad$ $\mathcal{A}(2)/\mathcal{A}(2)\{\mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^1\,\mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^4\}$

There are many other $\mathcal{A}(2)$-modules including many cyclic ones. Here is an interesting example that is realisable as the cohomology of a tmf-module.

$$\mathcal{A}(2)/\mathcal{A}(2)\{\mathrm{Sq}^1, A, B\}$$

$$A = \mathrm{Sq}^4\,\mathrm{Sq}^2 + \mathrm{Sq}^2\,\mathrm{Sq}^1\,\mathrm{Sq}^2\,\mathrm{Sq}^1,$$
$$B = \mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^4 + \mathrm{Sq}^1\,\mathrm{Sq}^2\,\mathrm{Sq}^1\,\mathrm{Sq}^2\,\mathrm{Sq}^4 + \mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1\,\mathrm{Sq}^2\,\mathrm{Sq}^1\,.$$



It doesn't come from an $\mathcal{A}$-module since Adem relations imply

$$\mathrm{Sq}^2\,\mathrm{Sq}^1\,\mathrm{Sq}^2\,\mathrm{Sq}^4\,x_0 = (\mathrm{Sq}^8\,\mathrm{Sq}^1 + \mathrm{Sq}^1\,\mathrm{Sq}^8)x_0.$$

Stable self duality

Many of the examples of modules we have seen are (*stably*) *self-dual*: A left module M over a graded Hopf algebra is *stably self-dual* if for some k, $\mathrm{D}M \cong M[k]$.
Every finite dimensional Hopf algebra is a Frobenius algebra or in the graded case a Poincaré duality algebra, hence stably self-dual. For example, as $\mathcal{A}(1)$-modules $\mathrm{D}\mathcal{A}(1) \cong \mathcal{A}(1)[-6]$, and as $\mathcal{A}(2)$-modules $\mathrm{D}\mathcal{A}(2) \cong \mathcal{A}(1)[-23]$.

Algebraic question: When is a cyclic module $\mathcal{A}(n)/L$ (where L is a left ideal) stably self-dual?
Partial answer: Any $\mathcal{A}(n)$-module of form $\mathcal{A}(n) \otimes_K \mathbb{F}_2$ for a subHopf algebra $K \subseteq \mathcal{A}(n)$. This is related to the idea of a *Frobenius extension* of Hopf algebras. If K is also a normal subalgebra then $\mathcal{A}(n)//K = \mathcal{A}(n) \otimes_K \mathbb{F}_2$ is a quotient Hopf algebra so this result is then immediate. If K is only a subalgebra then $\mathcal{A}(n) \otimes_K \mathbb{F}_2$ need not be stably self-dual.

There is a version of Spanier-Whitehead duality for finite CW R-modules and $H^*_R(D_R X) \cong \mathrm{D}(H^*_R(X))$ as left $H^*_R(H)$-modules. In particular, for a dualisable S-module W, $D_R(R \wedge W) \sim R \wedge D_S W$. A CW R-module Z is *stably self-dual* if $D_R Z \sim \Sigma^d Z$ for some d. There are many examples: If M is a compact closed n-manifold whose tangent or normal bundle is R-orientable then $D_R(R \wedge (M_+)) \sim R \wedge \Sigma^{-n}(M_+)$. So any Spin manifold satisfies $D_{k\mathrm{O}}(k\mathrm{O} \wedge (M_+)) \sim k\mathrm{O} \wedge \Sigma^{-n}(M_+)$ and any String manifold satisfies $D_{\mathrm{tmf}}(\mathrm{tmf} \wedge (M_+)) \sim \mathrm{tmf} \wedge \Sigma^{-n}(M_+)$.

A particular case of this occurs when $M = G$ is a compact connected Lie group and then $D(G_+) \sim \Sigma^{-n}(G_+)$. Here the suspension spectrum $\Sigma^\infty G_+$ is an S-algebra and its homology $H_*(G)$ is a Poincaré duality algebra over $\mathcal{A}$.

Here is a generalisation of this in the setting of 2-complete modules over connective examples such as $R = k\mathrm{O}$ and $R = \mathrm{tmf}$. Suppose that E is an R ring spectrum for which $P_* = H^R_* E$ is a local Poincaré duality algebra over $\mathbb{F}_2$. The Spanier-Whitehead dual of E satisfies

$$H^R_* D_R E \cong H^*_R E \cong H^R_* E[d]$$

as $H^*_R H$-modules.

Proposition

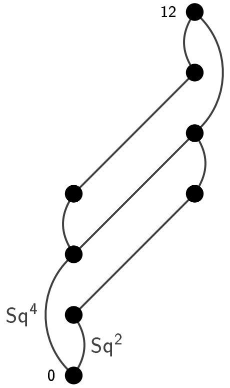
There is a morphism of R-modules $E \to \Sigma^d R$ which induces a non-trivial homomorphism

$$H^R_* E \to H^R_* R[-d] = \mathbb{F}_2[-d].$$

The multiplication map $E \wedge_R E \to E$ composed with this map define a duality pairing $E \wedge_R E \to \Sigma^d R$. Therefore E is a Spanier-Whitehead stably self-dual R-module with $D_R E \sim \Sigma^{-d} E$.

To see some exotic examples of this when $R = \mathrm{tmf}$, we can take $E = k\mathrm{O}, k\mathrm{U}, \mathrm{tmf}_1(3), H\mathbb{Z}, H\mathbb{F}_2$. For example, as an $\mathcal{A}(2)$-module, $H^*_{\mathrm{tmf}}(\mathrm{tmf}_1(3))$ is the double of $\mathcal{A}(1)$.

$$^{(1)}\mathcal{A}(1) = \mathcal{A}(2) \otimes_{\mathcal{E}(2)} \mathbb{F}_2$$



Joker modules: $\mathrm{tmf} \wedge (\mathrm{Joker})$ and $\mathrm{tmf} \wedge D(\mathrm{Joker})$ have different cohomology $\mathcal{A}(2)$-modules.
Homogeneous spaces G/H provide a good source of examples: e.g., $\mathrm{SU}(5)/\mathrm{SO}(5)$ is Spin, but $\mathrm{SU}(6)/\mathrm{SO}(6)$ is not.

Thanks for listening, keep safe and well!