Geodesic complexity of motion planning

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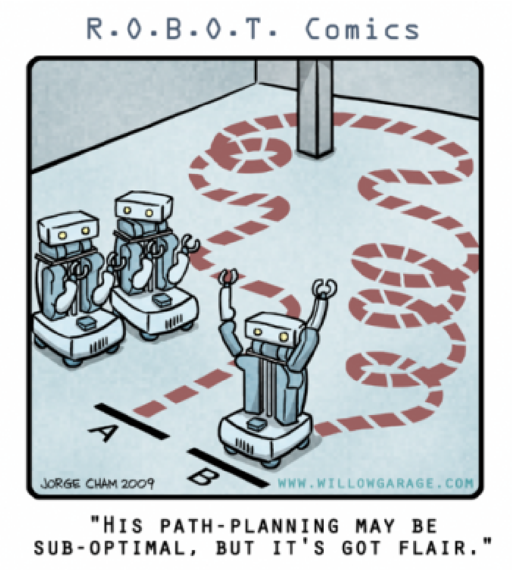
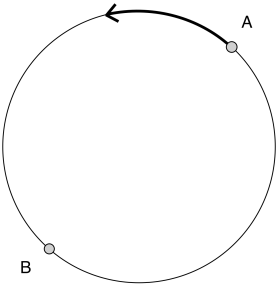
Motion planning problem



Motion planning on S^1



Two disjoint **continuous** rules to go from A to B:

- If A and B are not antipodal, then take the unique minimal geodesic.
- If A and B are antipodal, then take the minimal geodesic in anti-clockwise direction.

Motion planning on S^1

There does not exist a global **continuous** motion planning rule on the circle.

Proposition

There is a continuous motion planner on X if and only if X is contractible.

This means that the **minimum** number of continuous rules for any motion planner on the circle is 2.

Topological complexity

Preliminary definition

A continuous motion planner assigns to each pair of points on a space X a path between them in a continuous way.
In other words, it is a section of the **free path fibration**

$$PX \to X \times X \quad , \quad \gamma \mapsto (\gamma(0), \gamma(1)).$$

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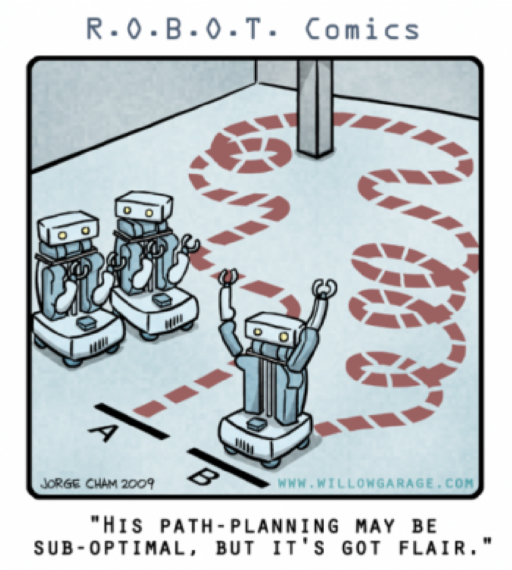
Definition (Farber '03) (Locally compact version)

The **topological complexity** $\mathrm{TC}(X)$ of a space X is the smallest k for which there exists a decomposition into locally compact sets (every point has a compact neighborhood)

$$X \times X = E_0 \cup \ldots \cup E_k, \qquad E_i \cap E_j = \emptyset \text{ if } i \neq j,$$

such that there exists a local section of the free path fibration over each E_i.

Motion planning problem



Geodesics

Definition

Let (X, d) be a metric space. The length of a path $\gamma\colon [0,1] \to X$ is given by

$$\ell(\gamma) = \sup_{0=t_0\leq t_1\leq \ldots \leq t_N=1} \sum_{i=1}^{N} \mathrm{d}(\gamma(t_{i-1}), \gamma(t_i)),$$

where the supremum is taken over all finite partitions of the interval $[0,1]$.

Definition

Let (X, d) be a metric space. We say that a path γ is a **shortest path** if $\ell(\gamma) = \mathrm{d}(\gamma(0), \gamma(1))$.

Geodesics

Definition

Let (X, d) be a metric space. We say that a path γ is a **geodesic** if there exists a number λ such that

$$\mathrm{d}(\gamma(t), \gamma(t')) = \lambda |t - t'|$$

for all $0 \leq t < t' \leq 1$.

Remark

Every geodesic is also a shortest path $\ell(\gamma) = \lambda = \mathrm{d}(\gamma(0), \gamma(1))$. We can think of geodesics as shortest paths with constant speed.
This definition of geodesic is often called a **minimizing geodesic** in differential geometry (for Riemannian manifolds). Geodesics in differential geometry are only locally length minimizing.

Geodesic complexity

Preliminary definition

Let (X, d) be a metric space. Let $GX \subset PX$ consist of the geodesics in X. Restricting the free path fibration to GX results in a map $GX \to X \times X$.

Geodesic complexity

Preliminary definition

Let (X, d) be a metric space. Let $GX \subset PX$ consist of the geodesics in X. Restricting the free path fibration to GX results in a map $GX \to X \times X$.

Definition

The **geodesic complexity** $\mathrm{GC}(X)$ of a metric space (X, d) is the smallest k for which there exists a decomposition into locally compact sets

$$X \times X = E_0 \cup \ldots \cup E_k, \qquad E_i \cap E_j = \emptyset \text{ if } i \neq j,$$

such that there exists a local section of $GX \to X \times X$ over each E_i.

Proposition (R.-M.)

If we defined GX to be the space of all **shortest paths** of X (instead of geodesics), we would get the same number $\mathrm{GC}(X)$.

Comparing TC and GC

Question

Clearly $\mathrm{TC}(X) \leq \mathrm{GC}(X)$, but are there cases in which $\mathrm{TC}(X) = \mathrm{GC}(X)$?

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Example

Because the motion planner we constructed for S^1 was geodesic $\mathrm{TC}(S^1) = \mathrm{GC}(S^1) = 1$ (two continuous rules).

Motion planning on S^2

To motion plan geodesically on S^2 we need to make a continuous choice of direction at every point of the sphere.

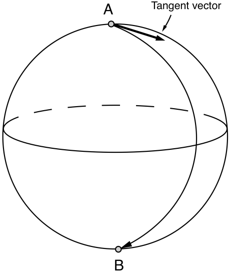

Figure: Geodesic in the direction of a tangent vector.

Problem with this approach

Hairy ball theorem

There does not exist a non-vanishing vector field on S^2.

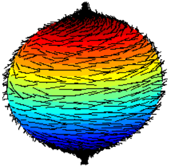

Figure: Hairy ball

Remark

However, there exists a vector field on S^2 with just with one zero.

Modify the approach slightly

Theorem

There exists a vector field on the sphere S^2 with *exactly one* zero.

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Choose a vector field with just one zero $X_0 \in S^2$.
There is a motion planner with three rules:

- If A and B are not antipodal, take the unique geodesic.

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Theorem

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Choose a vector field with just one zero $X_0 \in S^2$.
There is a motion planner with three rules:

- If A and B are not antipodal, take the unique geodesic.
- If A and B are antipodal and $A \neq X_0$, take the geodesic in the direction of the vector at A.

Modify the approach slightly

Theorem

There exists a vector field on the sphere S^2 with *exactly one* zero.

Choose a vector field with just one zero $X_0 \in S^2$.
There is a motion planner with three rules:

- If A and B are not antipodal, take the unique geodesic.
- If A and B are antipodal and $A \neq X_0$, take the geodesic in the direction of the vector at A.
- There is just one pair of points left: $A = X_0$ and $B = -X_0$. Choose a geodesic between X_0 and $-X_0$.

Modify the approach slightly

Theorem

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This implies $\mathrm{TC}(S^2) \leq 3$.

TC of spheres

Theorem (Farber '03)

$$\mathrm{TC}(S^n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \geq 2 \text{ is even} \end{cases}$$

Corollary

Because the optimal motion planners given by Farber are geodesic:

$$\mathrm{GC}(S^n) = \mathrm{TC}(S^n)$$

TC of projective spaces

Theorem (Farber–Tabachnikov–Yuzvinsky '03)

$$\mathsf{TC}(\mathbb{R}P^n) = \begin{cases} n & \text{if } n = 1, 3, 7 \\ \mathsf{Immdim}(\mathbb{R}P^n) & \text{otherwise} \end{cases}$$

Corollary

Because the motions planners given by Farber–Tabachnikov–Yuzvinsky can be modified to be geodesic:

$$\mathsf{GC}(\mathbb{R}P^n) = \mathsf{TC}(\mathbb{R}P^n)$$

Comparing TC and GC

Question

We just saw that in some cases $\mathrm{TC}(X) = \mathrm{GC}(X)$. Can we find a metric space X such that $\mathrm{TC}(X) < \mathrm{GC}(X)$?

Convex subspaces

Definition

A subspace Y of a metric space X is said to be **convex** if for any pair of points $x, y \in Y$, every minimal geodesic in X between x and y lies entirely in Y.

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Lemma (R.-M.)

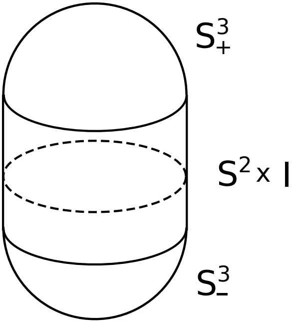
If Y is a locally compact convex subspace of X, then $\mathrm{TC}(Y) \leq \mathrm{GC}(Y) \leq \mathrm{GC}(X)$.

Elongated 3-sphere

Example

Let $\tilde{S}^3$ be the result of glueing two caps on the cylinder $S^2 \times I$.
Clearly every geodesic motion planner on $\tilde{S}^3$ restricts to a motion planner on S^2. Therefore:

$$\mathrm{GC}(\tilde{S}^3) \geq \mathrm{TC}(S^2) = 2 > 1 = \mathrm{TC}(S^3) = \mathrm{TC}(\tilde{S}^3).$$



Convex subspaces

Theorem (R.-M.)

For every $k \in \mathbb{N}$ there exists a closed Riemannian manifold (M, g) such that $\mathrm{GC}(M) - \mathrm{TC}(M) \geq k$. In fact, M can be chosen to be a sphere (with a non-standard metric). Furthermore, for every $k \in \mathbb{N}$, there exists a metric g'_m on $\mathbb{R}^{k+1}$ such that $\mathrm{GC}(\mathbb{R}^{k+1}, g'_m) \geq k$, while $\mathrm{TC}(\mathbb{R}^{k+1}) = 0$.

Remark

This shows that $\mathrm{GC}(M)$ is very different from the efficient topological complexity $\ell TC(M)$ of Błaszczyk–Carrasquel, for which they show that $TC(M) \leq \ell TC(M) \leq TC(M) + 1$ if M is a closed Riemannian manifold.

Total cut locus

Definition

Let (X, d) be a metric space. The **total cut locus** of X is the subset $C \subset X \times X$ consisting of all pairs (x, y) for which there is more than one geodesic γ from x to y. The **cut locus of a point** $x \in X$ is the subset $C_x \subset X$ consisting of all y in X such that (x, y) is in C.

Total cut locus

Theorem (R.-M.)

Let (X, d) be a locally compact complete metric space such that X is a geodesic space (every pair of points $x, y \in X$ is connected by a geodesic). Then the map $\pi\colon GX \to X \times X$ has a local section over the complement of the total cut locus. In particular, if $C = \emptyset$ then $\mathrm{GC}(X) = 0$.

Remark

It is worth noting that $\mathrm{GC}(X) = 0$ does not imply that $C = \emptyset$. For example, the square $I \times I$ equipped with the Manhattan metric $d_1((x, y)(x', y')) = |x - x'| + |y - y'|$ has $\mathrm{GC}(I \times I) = 0$ but C consists of all pairs $((x, y)(x', y'))$ such that $x \neq x'$ and $y \neq y'$.

Klein bottle

Theorem (Cohen–Vandembroucq '18)

Let K be the Klein bottle. Then $\mathrm{TC}(K) = 4$.

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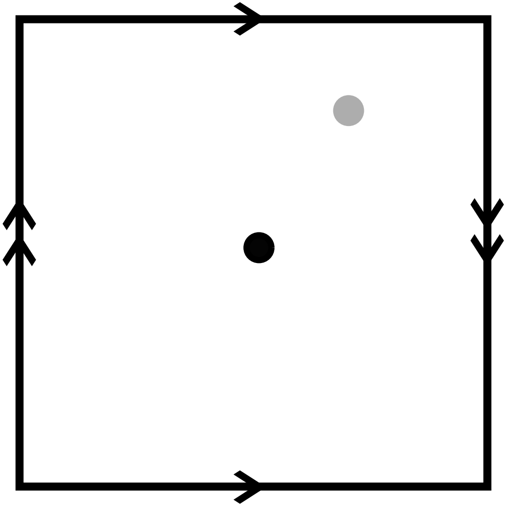
Let K be the Klein bottle. Then $\mathrm{TC}(K) = 4$.

Theorem (R.-M.)

Let K_{flat} be the flat Klein bottle. Then $\mathrm{GC}(K_{flat}) = 4$.

Although the lower bound follows from $\mathrm{GC}(K_{flat}) \geq \mathrm{TC}(K) = 4$, we show the **lower bound** directly by studying the **total cut locus**.

Motion planning on K_{flat}



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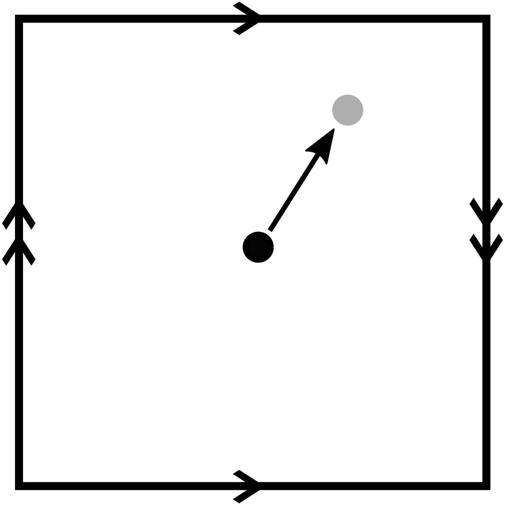

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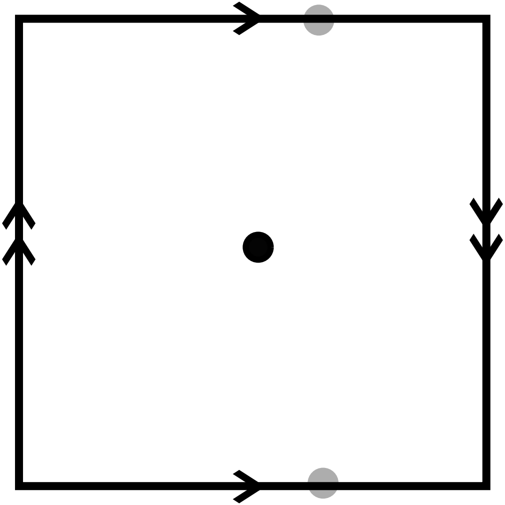

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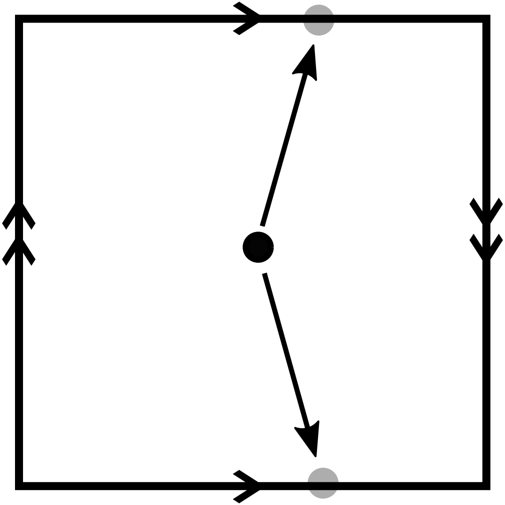

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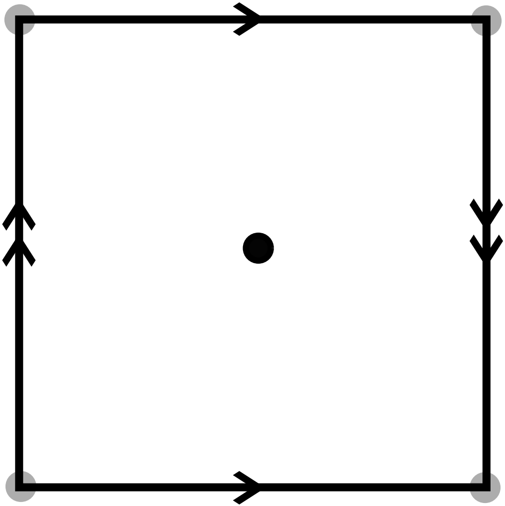

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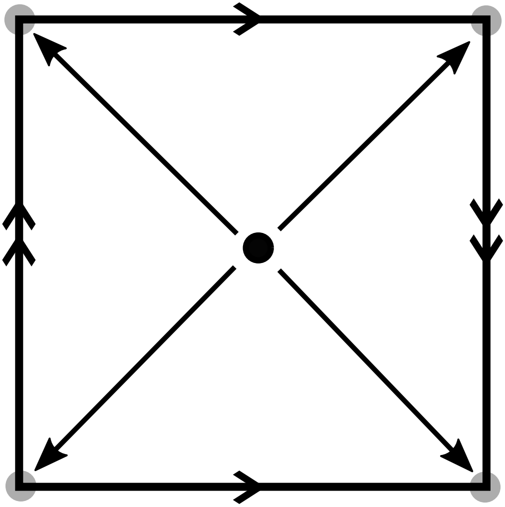

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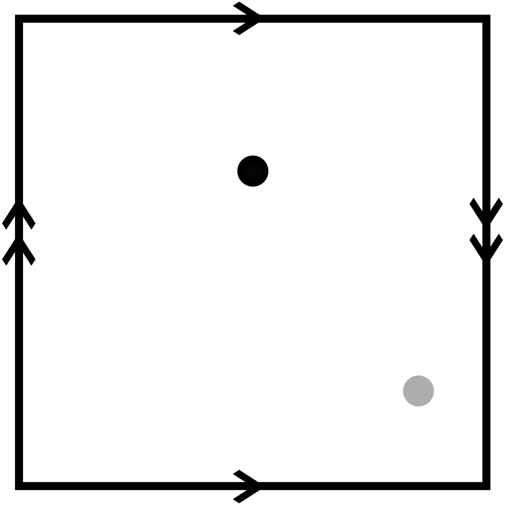

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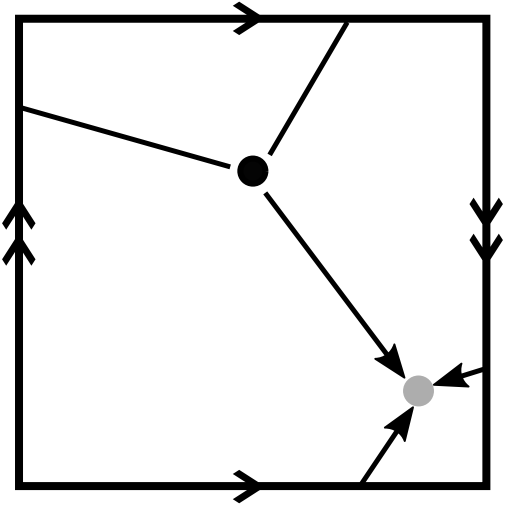


Proof of GC(K_{flat}) = 4

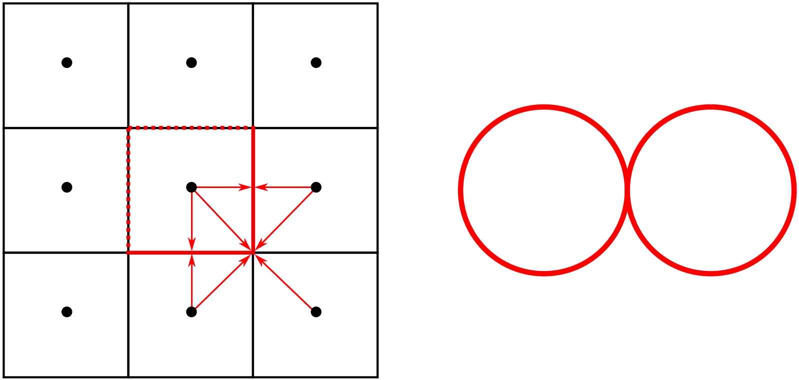


Figure: Cut locus of $x = (1/2, 1/2)$ in the Klein bottle.

Proof of $GC(K_{flat}) = 4$

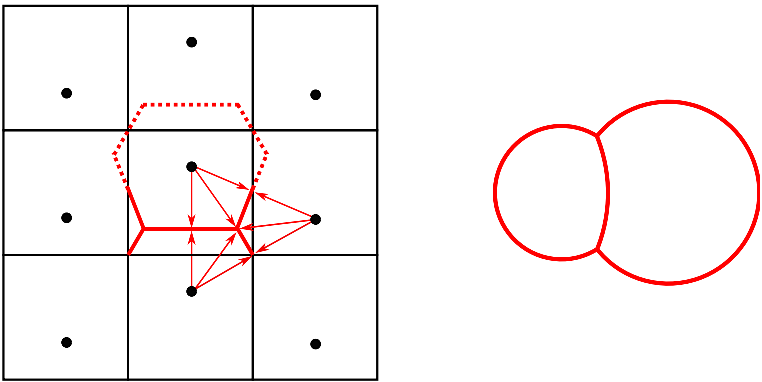


Figure: Cut locus of x going "up" from $(1/2, 1/2)$ to $(1/2, 1)$ in the Klein bottle. When x moves away from $(1/2, 1/2)$ a new edge appears at the vertex and then it keeps growing, while another edge gets shorter.

Proof of $\mathrm{GC}(K_{flat}) = 4$

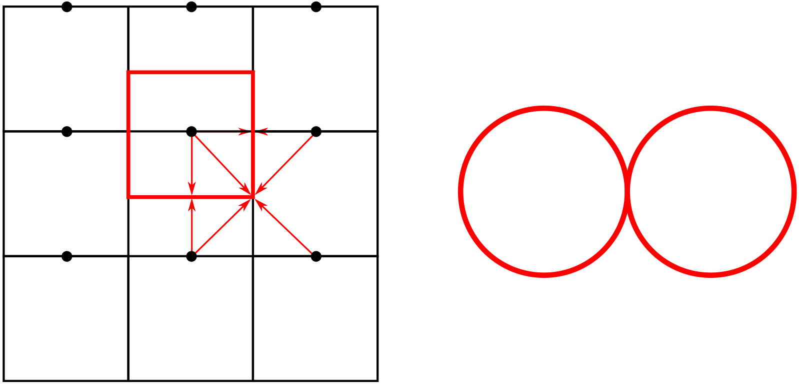


Figure: Cut locus of $x = (1/2, 1)$ in the Klein bottle. The new edge that appeared when moving up from $(1/2, 1/2)$ has completely replaced the old vertical edge.

Lower bound for Klein bottle

Let $S_k \subset K_{flat} \times K_{flat}$ consist of all pairs (x, y) such that there are precisely k minimal geodesics from x to y. Over each S_k the map $GK_{flat} \to K_{flat} \times K_{flat}$ restricts to a k-sheeted covering.

There are exactly 4 sets S_k. By studying how those coverings fit together, we can show that there aren't geodesic motion planners with fewer than 5 sets.

Higher Klein bottles

Using the same ideas (but applied to much more complicated total cut loci), we were able to compute $\mathrm{GC}(K_n)$ of all higher Klein bottles K_n, in joint work with Donald Davis.

Definition (Davis)

The n-dimensional Klein bottle is given by

$$K_n = (S^1)^n/(z_1, \ldots, z_{n-1}, z_n) \sim (\bar{z}_1, \ldots, \bar{z}_{n-1}, -z_n).$$

Theorem (Davis–R.-M.)

The geodesic complexity of the flat higher Klein bottles K_n is given by $\mathrm{GC}(K_n) = 2n$.

The topological complexity of K_n is unknown, except for the Klein bottle K_2.

Tori

Theorem (R.M.) (TC result due to Farber)

Let T^n_{flat} be the n-torus equipped with the flat metric. Then

$$\mathrm{GC}(T^n_{flat}) = \mathrm{TC}(T^n) = n.$$

Theorem (R.M.)

Let T^2_{emb} be the embedded torus. Then

$$\mathrm{GC}(T^2_{emb}) = 3 > \mathrm{GC}(T^2_{flat}) = 2.$$

Flat sphere

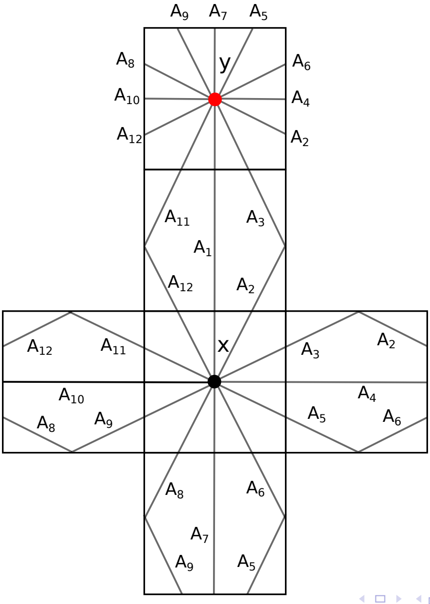
Definition

Let W^2 be the boundary of a cube with the flat metric. We may call it a **flat sphere**. This example was suggested by Jarek Kędra.

Theorem (R.-M.)

$$\mathrm{GC}(W^2) \geq 3 > \mathrm{TC}(W^2) = \mathrm{TC}(S^2) = 2$$

Potential geodesics between opposite faces



Flat sphere

- $L_1^2 = (x_1 - y_1)^2 + (2 - x_2 + y_2)^2$
- $L_2^2 = (1 - x_1 + y_2)^2 + (2 - x_2 - y_1)^2$
- $L_3^2 = (1 - x_2 - y_1)^2 + (2 - x_1 + y_2)^2$
- $L_4^2 = (x_2 + y_2)^2 + (2 - x_1 - y_1)^2$
- $L_5^2 = (1 + x_2 - y_1)^2 + (2 - x_1 - y_2)^2$
- $L_6^2 = (1 - x_1 - y_2)^2 + (2 + x_2 - y_1)^2$
- $L_7^2 = (x_1 - y_1)^2 + (2 + x_2 - y_2)^2$
- $L_8^2 = (1 + x_1 - y_2)^2 + (2 + x_2 + y_1)^2$
- $L_9^2 = (1 + x_2 + y_1)^2 + (2 + x_1 - y_2)^2$
- $L_{10}^2 = (x_2 + y_2)^2 + (2 + x_1 + y_1)^2$
- $L_{11}^2 = (1 - x_2 + y_1)^2 + (2 + x_1 + y_2)^2$
- $L_{12}^2 = (1 + x_1 + y_2)^2 + (2 - x_2 + y_1)^2$

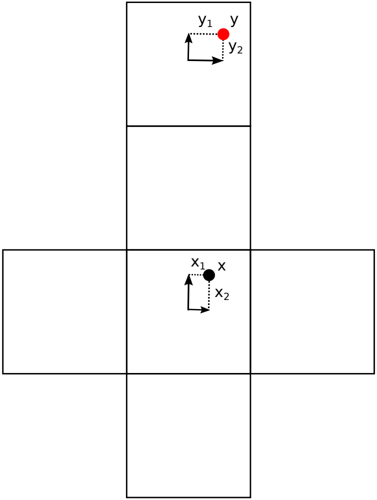


Figure: Coordinates for points on opposite faces.

Flat sphere

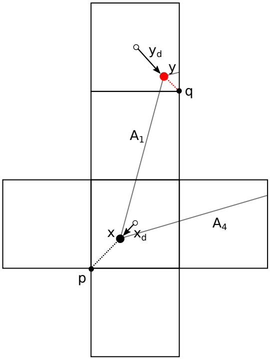
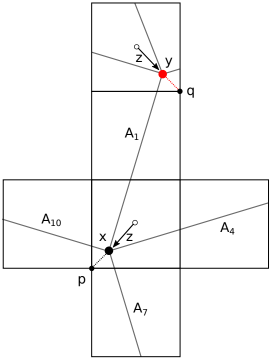


Figure: Geodesics on non-symmetric diagonals.



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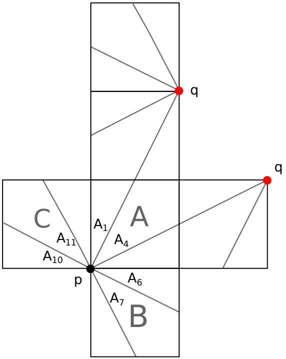

Figure: Geodesics between opposite corners.

Ordered configuration spaces on graphs

Consider the graph G with path metric d:

Definition

The **ordered two-point ε-configuration space** is

$$\mathrm{F}_\varepsilon := F_\varepsilon(G,2) = \left\{(a_1,a_2) \in G \times G \;\middle|\; d(a_1,a_2) \geq \varepsilon\right\},$$

Ordered configuration spaces on graphs

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Theorem (Davis–Harrison–R.-M.) (TC results due to Farber)

Let G be a star graph, with $k \geq 3$ edges emanating from a single vertex, and let F_ε be the ordered two-point ε-configuration space of G. Then

1. If $k = 3$, $\mathrm{GC}(\mathrm{F}_\varepsilon, \ell_i) = \mathrm{TC}(\mathrm{F}_\varepsilon) = 1$, for $i \in \{1,2\}$.
2. If $k \geq 4$, $\mathrm{GC}(\mathrm{F}_\varepsilon, \ell_i) = \mathrm{TC}(\mathrm{F}_\varepsilon) = 2$, for $i \in \{1,2\}$.

$$\ell_1(a,b) = |d(a_1,b_1) + d(a_2,b_2)|,$$
$$\ell_2(a,b) = \sqrt{d(a_1,b_1)^2 + d(a_2,b_2)^2}.$$

Unordered configuration spaces on graphs

Definition

The **unordered two-point configuration space** is

$$\mathrm{C} := C(G,2) = F(G,2)/\mathbb{Z}_2$$
$$= \{(a_1,a_2) \in G \times G \mid a_1 \neq a_2\} /[(a_1,a_2) \sim (a_2,a_1)].$$

Unordered configuration spaces on graphs

Definition

The **unordered two-point configuration space** is

$$\begin{aligned} \mathrm{C} &:= C(G,2) = F(G,2)/\mathbb{Z}_2 \\ &= \{(a_1,a_2) \in G \times G \mid a_1 \neq a_2\}/[(a_1,a_2) \sim (a_2,a_1)]. \end{aligned}$$

Theorem (Davis–Harrison–R.-M.) (TC results due to Farber)

Let G be a tree and let C be the unordered two-point configuration space of G. Then

1. If G is homeomorphic to an interval, then $\mathrm{GC}(\mathrm{C},\ell_1) = \mathrm{TC}(\mathrm{C}) = 0$.
2. If G is the Y graph, $\mathrm{GC}(\mathrm{C},\ell_1) = \mathrm{TC}(\mathrm{C}) = 1$.
3. Otherwise, $\mathrm{GC}(\mathrm{C},\ell_1) = \mathrm{TC}(\mathrm{C}) = 2$.

Thank you!