

Configuration spaces of surfaces

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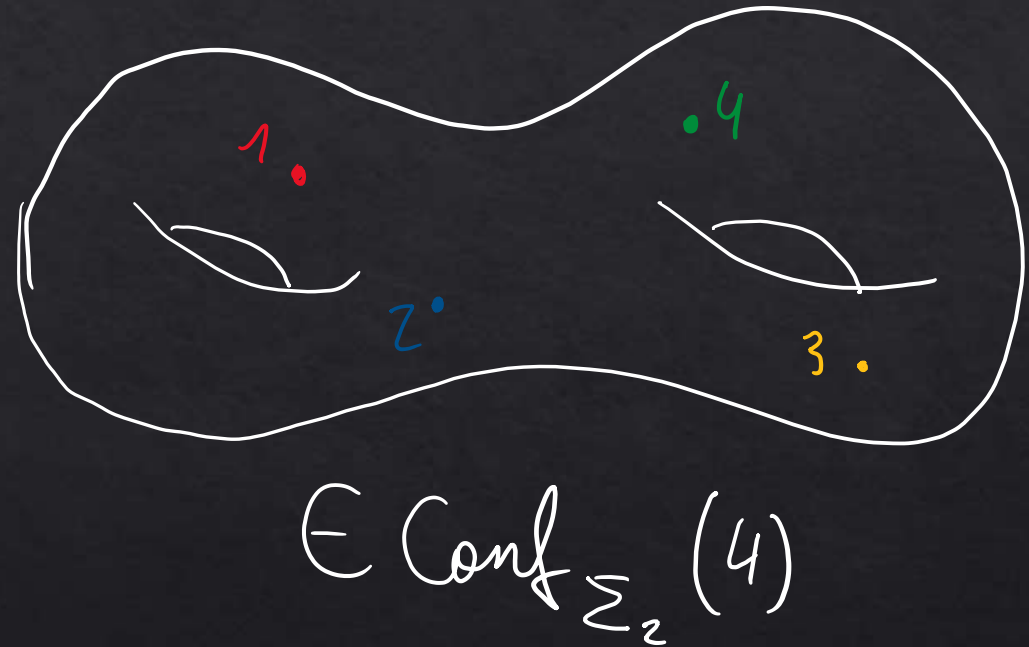
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Configuration spaces

- ◆ Let M be a manifold.
- ◆ $\text{Conf}_M(r) := \{ (x_1, \dots, x_r) \in M^r \mid \forall i \neq j, \quad x_i \neq x_j \}$
- ◆ Classical objects in algebraic topology.
- ◆ First appearance: Hurwitz, 1891.

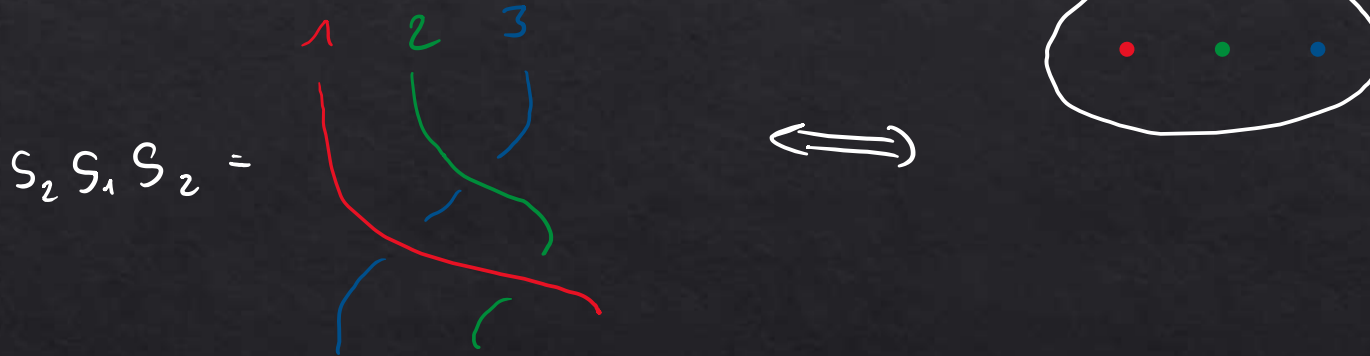


Braid groups

- First classical appearance: braid groups

$$B_r := \langle s_1, \dots, s_{r-1} \mid s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}; \quad |i - j| \geq 2 \Rightarrow s_i s_j = s_j s_i \rangle.$$

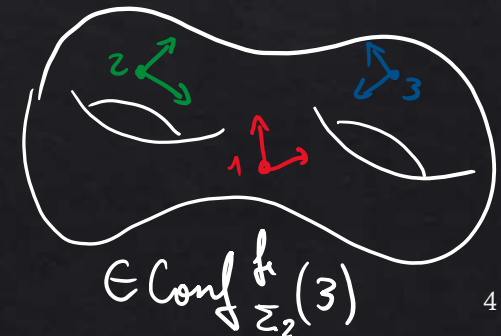
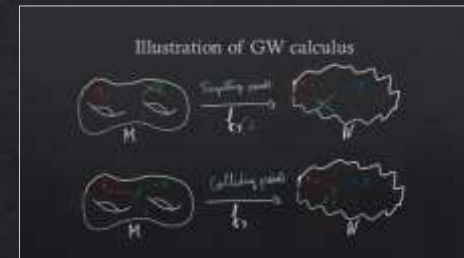
- Element of B_r = path in $\text{Conf}_{D^2}(r)$:



- B_r is $\pi_1(\text{Conf}_{D^2}(r) / \mathfrak{S}_r)$, and $PB_r = \ker(B_r \rightarrow \mathfrak{S}_r)$ is $\pi_1(\text{Conf}_{D^2}(r))$.
- Eilenberg-MacLane spaces \Rightarrow anything about $(P)B_r$ can be computed from $\text{Conf}_{D^2}(r)$.

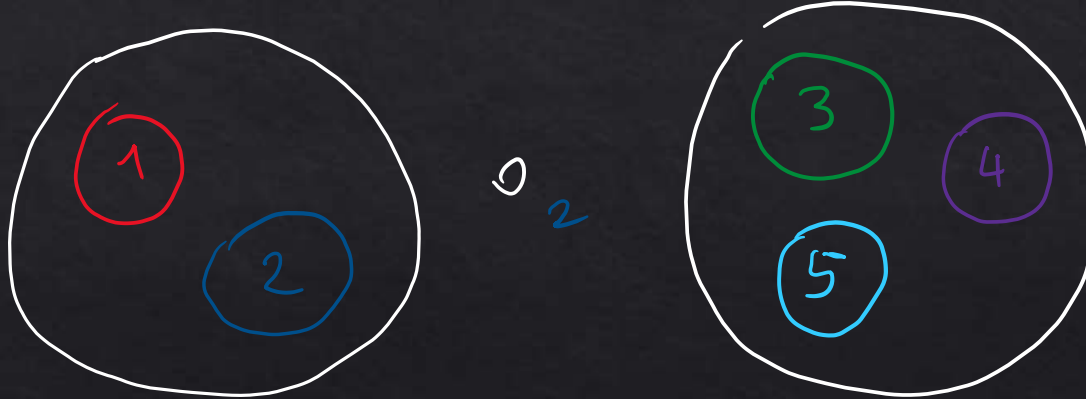
Goodwillie–Weiss manifold calculus

- Want to compute $\text{Emb}(M, N) = \{ f: M \hookrightarrow N \mid f \text{ is an embedding} \}$.
- $\text{Emb}(M, N)$ is a subspace of $\text{Map}_{\mathfrak{S}}(\text{Conf}_M, \text{Conf}_N) = \prod_{r=0}^{+\infty} \text{Map}_{\mathfrak{S}_r}(\text{Conf}_M(r), \text{Conf}_N(r))$.
- GW calculus “approximates” $\text{Emb}(M, N)$ by a more easily computable subspace:
- Collections of maps $(f_r)_{r \geq 0} \in \text{Map}_{\mathfrak{S}}(\text{Conf}_M, \text{Conf}_N)$ such that:
 - Forgetting in the source maps to forgetting in the target *up to homotopy*;
 - Proximity in the source maps to proximity in the target *up to homotopy*.
- Restrict to $\text{Map}_{\mathfrak{S}}(\text{Conf}_M^{\text{fr}}, \text{Conf}_N^{m-\text{fr}})$ to have a correct homotopy type.
- If $\dim N - \dim M \geq 3 \Rightarrow$ recover the homotopy type of $\text{Emb}(M, N)$.



Operadic structure

- ◊ We want to clarify what “compatible up to homotopy” means
- ◊ \Rightarrow we need operads!
- ◊ Let $D_M^{\text{fr}}(r) := \text{Emb}(\coprod_{i=1}^r \mathbb{D}^m, M)$ and $D_n^{\text{fr}}(r) := \text{Emb}(\coprod_{i=1}^r \mathbb{D}^m, \mathbb{D}^m)$
- ◊ $D_m^{\text{fr}} := \{D_m^{\text{fr}}(r)\}_{r \geq 0}$ is the (framed) **little disks operad**:



Extra structure:

$$D_m^{\text{fr}}(r) \times D_m^{\text{fr}}(s) \rightarrow D_m^{\text{fr}}(r + s - 1)$$

- ◊ $D_M^{\text{fr}} := \{D_M^{\text{fr}}(r)\}_{r \geq 0}$ is a **right module** over D_m^{fr} via $D_M^{\text{fr}}(r) \times D_m^{\text{fr}}(s) \rightarrow D_M^{\text{fr}}(r + s - 1)$

Operads & GW calculus

- Any embedding $f : M \hookrightarrow N$ induces a **morphism** $D_M^{\text{fr}} \rightarrow D_N^{\text{fr}}$, not just a map between configuration spaces!

- Theorem** [Arone, Boavida, Goodwillie, Sinha, Turchin, Weiss ...].

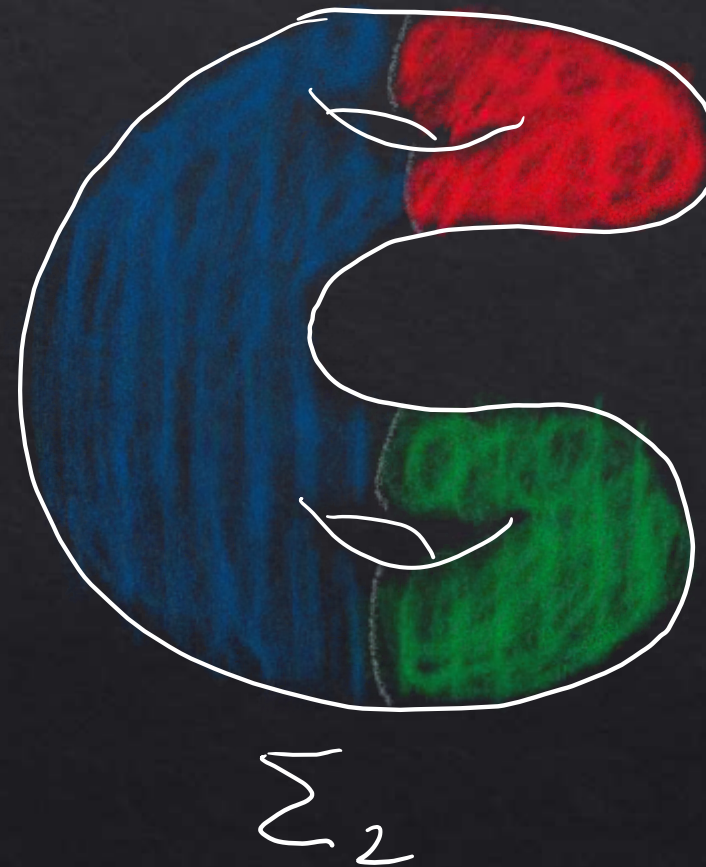
If $\dim N - \dim M \geq 3$, then

$$\text{Emb}(M, N) \simeq \mathbb{R}\text{Map}_{D_m^{\text{fr}}}(D_M^{\text{fr}}, D_N^{m-\text{fr}}).$$

- Upshot: if we know the homotopy type of the **collection** of configuration spaces as **right modules**, then we can compute embedding spaces.
- Computing the homotopy type of $\text{Conf}_M(r)$ is difficult. For example,
 $M \simeq M' \not\Rightarrow \text{Conf}_M(r) \simeq \text{Conf}_{M'}(r).$

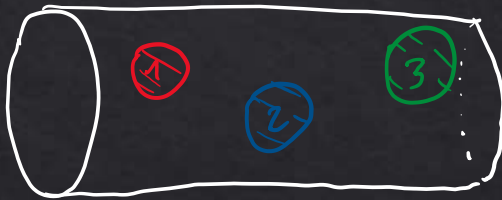
Approach: cut the surface

- ◆ Take $\Sigma_g = (S^1 \times S^1) \# \dots \# (S^1 \times S^1)$;
- ◆ Cut! $\Sigma_g = (S^2 \setminus (D^2)^{\sqcup 2g}) \cup (S^1 \times \mathbb{R})^{\sqcup g}$.
- ◆ Each part is $D^2 \setminus \mathbf{k}$ for some k .
- ◆ We have a fiber bundle:
 $\text{Conf}_{M \setminus *}^{\text{fr}}(r) \rightarrow \text{Conf}_M^{\text{fr}}(r+1) \rightarrow \text{Fr}_M$
 \rightarrow computation by **induction**.
- ◆ We just need to know $\text{Conf}_{D^2 \setminus k}^{\text{fr}}(r)$.



Hochschild complex

- ◆ The collection $D_{N \times \mathbb{R}}^{\text{fr}} = \{D_{N \times \mathbb{R}}^{\text{fr}}(r)\}_{r \geq 0}$ is a monoid up to homotopy:



$$D_{N \times \mathbb{R}}^{\text{fr}}(3) \times D_{N \times \mathbb{R}}^{\text{fr}}(2) \longrightarrow D_{N \times \mathbb{R}}^{\text{fr}}(5)$$

- ◆ If $\partial M = N$, then D_M^{fr} is a left module over $D_{N \times \mathbb{R}}^{\text{fr}}$.

- ◆ We have:

$$D_{M \cup N \times \mathbb{R} M'}^{\text{fr}} \simeq D_M^{\text{fr}} \otimes_{D_{N \times \mathbb{R}}^{\text{fr}}}^{\mathbb{L}} D_{M'}^{\text{fr}}.$$

- ◆ Upshot: $D_{\Sigma_g}^{\text{fr}}$ is an “iterated Hochschild complex” of the $(D_{S^1 \times \mathbb{R}}^{\text{fr}})^{\otimes g}$ -bimodule $D_{S^2 \setminus 2g}^{\text{fr}}$.

Rational homotopy theory

- ◇ The whole homotopy type is too complex.
- ◇ We focus on **characteristic zero**.
- ◇ **Definition:** $f : X \rightarrow Y$ is a **rational equivalence** if
$$\pi_*(f) \otimes_{\mathbb{Z}} \mathbb{Q} : \pi_*(X) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \pi_*(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$$
is an isomorphism.
- ◇ **Theorem** [Sullivan]: There is an equivalence $\Omega^* \dashv \langle - \rangle$ between:
 - ◇ Simply connected finite-type spaces, up to rational equivalence;
 - ◇ Simply connected finite-type commutative differential-graded algebras, up to quasi-isomorphism.
- ◇ Upshot: we want to find a **model** of $\Omega^*(D_M^{\text{fr}})$ with its action of $\Omega^*(D_m^{\text{fr}})$.

Formality

- ◆ **Theorem** [Kontsevich, Tamarkin, Lambrechts–Volić,] The operad D_2 is **formal**, i.e.,

$$H^*(D_2; \mathbb{Q}) \simeq \Omega^*(D_2).$$

- ◆ \Rightarrow we know everything about $D_2^{\mathbb{Q}}$ from [Arnold, Cohen]:

$$H^*(D_2(r); \mathbb{Q}) = \frac{S(\omega_{ij})_{1 \leq i \neq j \leq r}}{(\omega_{ij}^2 = \omega_{ji} - \omega_{ij} = \omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij} = 0)}, \quad \deg \omega_{ij} = 1.$$

- ◆ Many important consequences, e.g., deformation quantization, Deligne conjecture...

Formality: two approaches

- ◇ Kontsevich's approach:
 - ◇ Replace the 3T relation by “internal vertices”;
 - ◇ Prove combinatorially that we have a resolution of $H^*(D_n)$;
 - ◇ Use integrals to connect with $\Omega^*(D_n)$.
- ◇ Generalized by Giansiracusa–Salvatore to prove formality of D_2^{fr} .
- ◇ Tamarkin's approach:
 - ◇ Find a simpler groupoid $\text{PaB} \simeq \pi D_2$;
 - ◇ Find a (Koszul) resolution of $H^*(D_2)$, the Drinfeld–Kohno Lie algebra;
 - ◇ Connect the two with a Drinfeld associator.
- ◇ Generalized by Ševera to prove formality of D_2^{fr} .
- ◇ **Theorem** [CIW] Cyclic formality of D_2^{fr} . Proof inspired by Ševera's.

The result

Theorem [CIW]. We have a small, explicit model $G_{\Sigma_g}^{\text{fr}}$ of $D_{\Sigma_g}^{\text{fr}}$, in arity r :

- ◇ Generators: ω_{ij} for $1 \leq i \neq j \leq r$; $\alpha_{1,i}, \dots, \alpha_{g,i}, \beta_{1,i}, \dots, \beta_{g,i}$ for $1 \leq i \leq r$; θ_i for $1 \leq i \leq r$.
- ◇ Relations:
 - ◇ Same as before: $\omega_{ij}^2 = \omega_{ji} - \omega_{ij} = \omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij} = 0$;
 - ◇ $\alpha_{k,i}\beta_{k,i} = \alpha_{l,i}\beta_{l,i}$ (volume form of Σ_g) and 0 otherwise;
 - ◇ Symmetry: $\alpha_{k,i}\omega_{ij} = \alpha_{k,j}\omega_{ij}, \beta_{k,i}\omega_{ij} = \beta_{k,j}\omega_{ij}, \theta_i\omega_{ij} = \theta_j\omega_{ij}$.
- ◇ Differential: $d\omega_{ij} = \Delta_{ij}$ and $d\theta_i = (2 - 2g) \cdot \text{vol}_i$.
- ◇ Proof: $G_{\Sigma_g}^{\text{fr}} \xleftarrow{\text{Combin.}} \text{Graphs}_{\Sigma_g}^{\text{fr}} \xrightarrow{\text{K}} \text{IterHoch} \left(H^* \left(D_{S^2 \setminus 2g}^{\text{fr}} \right); H^* \left(D_{S^1 \times \mathbb{R}}^{\text{fr}} \right) \right) \xleftarrow{\text{T}} \Omega^* \left(D_{\Sigma_g}^{\text{fr}} \right).$

Thank you for your attention!

These slides, the paper: <https://idrissi.eu>