Algebraic K theory of Lawvere Theories

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Algebraic K-theory:
originally invariant of rings

$$R \longmapsto K_*(R) = \pi_*(K(R))$$

space or spectrum

Rich invariant:
connections to number theory, alg. geom, etc.

Today K-theory as invariant of more general alg. structure:
Lawvere theories.

1.) What is a Lawvere theory?

2.) What is its K-theory?

3.) What can we say about it?

1.) Lawvere theories:
way of encoding algebraic structure
(cf. operads, monads)

Essentially: specify algebraic structure

by specifying the "free" obj.
with that structure on n generators
for each n.

<u>Ex</u>: • R ring. Cat. of free R-modules
$R^{\oplus n}$ for $n \geq 0$

$\rightsquigarrow$ Lawvere theory of R-mod.

• G group. Cat. of free G-sets $\coprod_n G$
$\rightsquigarrow$ Lawvere theory of G-sets.

$\rightsquigarrow$ Key subex: G trivial
$\rightsquigarrow$ Lawvere theory of sets.

More formally: E skeleton of cat. of fin. sets.

A <u>Lawvere theory</u> T is $(\mathbb{F}_T, E \to \mathbb{F}_T)$
where · $\mathbb{F}_T$ is cat. w/ (fin) coproducts.

· $E \to \mathbb{F}_T$ is a coprod. preserving functor that is bijective on objects.

(E, id) = Lawvere theory of sets.

<u>Ex</u>: Groups: $\mathbb{F}_{\text{Groups}}$ = (skeleton of) cat. of

groups   free groups.

- Presheaves on $\mathbb{F}_T$ taking $+ \longmapsto \times$
  ⇝ models for T.

2.) K-theory: Define K-theory of Lawvere theory T by

$$K(T) = K(\mathbb{F}_T^{\times})$$

K theory spectrum of the symm. mon. subcat. of $\mathbb{F}_T$ on the isos.

Familiar examples:

- $T =$ R-modules: $K(T) =$ (free mod variant of) $K(R)$
- $T =$ G-set $K(G\text{-set}) \simeq \Sigma^\infty_+ BG$ (Segal)

  $K(E) = \mathbb{S}$ sphere spectrum
- $T =$ Groups $K(\text{Groups}) = \mathbb{S}$ (Galatius)

Where do these come from?
connection to automorphism groups:

Lawvere theory T, write T_n = image of $n \in E$

Get sequence

$$\cdots \to \mathrm{Aut}(T_n) \longrightarrow \mathrm{Aut}(T_{n+1}) \to \cdots$$

$\rightsquigarrow$ stable automorphism group

$$\operatorname{colim}_n \mathrm{Aut}(T_n)$$

Ex: $GL_n(R)$ $\quad$ $\mathrm{Aut}(F_n)$

Thm (B-Szymik) There is an equivalence of spaces

$$\Omega^\infty K(T) \simeq K_0(T) \times B\operatorname{colim}_n \mathrm{Aut}(T_n)^{+}$$

& an iso. of groups

$$\operatorname{colim}_n H_*(\mathrm{Aut}(T_n)) \cong H_*(\Omega^\infty_0 K(T))$$

stable homology of autom. grps

0th comp.

$\to$ stable homology calculations can be viewed as calculations of K-theory of Lawvere theories.

- Galatius of free groups.
- Szymik-Wahl on Higman-Thompson

groups

- Comes from Quillen + construction approach to K-theory
- Use this approach to show invariance:

<u>Thm</u> (B-Szymik)

$$K(T) \simeq K(M_n(T))$$

$n \times n$ "matrix theory" on T

For a ring R, $M_n(R\text{-mod}) - M_n(R)\text{-mod}$

$$K(R) \simeq K(M_n(R))$$

which follows from Morita invariance of K-theory of rings.

There is a notion of Morita equiv. for Lawvere theories + $T \underset{\text{Morita}}{\simeq} M_n(T)$

But K-theory not fully Morita invariant!

There is a family of Morita equiv. Lawvere theories Post_v for $v = 1, 2, 3, \ldots$ with Post_2 = Boolean algebras.

(arise in logic)

<u>Thm</u> (B-Szymik):

$$K_*(\mathrm{Post}_v) = \pi_*(\mathbb{S}) \big/ {v\text{-power torsion}}.$$

In particular, K-theory not Morita invariant!

Another approach to K-theory is via Segal's construction:

<u>Thm</u> (Elmendorf-Mandell) Segal's

$$\mathrm{SymMon} \xrightarrow{K} \mathrm{Spectra}$$

is "multiplicative".

You'd like this to mean something like "is a lax monoidal functor" but that's not strictly possible because there's no appropriate $\otimes$ on SymMon.

But there is a $\otimes$ of Lawvere theories.

<u>Thm</u> (B-Szymik) composite

$$\begin{array}{ccccc} \mathrm{Lawvere} & \longrightarrow & \mathrm{SymMon} & \longrightarrow & \mathrm{Spectra} \\ T & \longmapsto & \mathbb{F}_T^{\times} & \longmapsto & K(\mathbb{F}_T^{\times}) \end{array}$$

is lax symm. mon. functor.

In particular, if S, T are Lawvere theories

$$K(S) \wedge K(T) \longrightarrow K(S \otimes T)$$

Kronecker product.

<u>Assembly</u>: $S = \mathbb{Z}$-module

$S \otimes T \simeq$ linearization of T
$\mathbb{Z}[T]$-module

if $T = G$-sets, $\mathbb{Z}[T] = \mathbb{Z}[G]$

Map above:

$$K(\mathbb{Z}) \wedge K(T) \longrightarrow K(\mathbb{Z}[T])$$

assembly for T

<u>Ex</u>: $T = G$-sets

$$K(\mathbb{Z}) \wedge \Sigma^\infty_+ BG \longrightarrow K(\mathbb{Z}[G])$$

Major question about assembly: is it an equiv?

<u>Thm</u> (B-Szymik): Assembly is an equiv. for the Lawvere theory of Cantor algebras of arity a
(i.e. sets w/ bijection $X^a \to X$)
$\rightsquigarrow$ Higman-Thompson group

Szymik-Wahl: $K(\mathrm{Cantor}_a) \simeq \mathbb{S}/(a-1)$

Thm (B-Szymik): Assembly for Boolean algebras is zero. (target is contractible).